Exponentiated Akash distribution and its applications

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In this study, a new distribution called Exponentiated Akash distribution, which is the extension of the one-parameter Akash distribution, is derived using the exponentiation technique. Statistical properties, such as; moments, moment generating function, reliability function, hazard function, Rényi entropy and order statistics of the distribution were obtained. A discussion was made based on the maximum likelihood procedure of finding estimates of parameters of the distribution. Our numerical illustrations substantiate the applicability, flexibility and the tendency of the distribution to provide better fits to certain sets of data than competing distributions.

Keywords: Akash distribution, Exponentiation, Order statistics, Quantile function, Reliability function.

1. Introduction

In order to improve the modeling of lifetime data in reliability analysis, Shanker (2015) introduced a new distribution known as Akash distribution. He argued the importance of this distribution and its superiority to other notable lifetime distributions such as the Lindley, Weibull and Exponential distributions, among others, using real life data sets. These distributions are generally referred to as baseline distributions when compared with generalized distributions. Baseline distributions are mostly generalized using the exponentiation method introduced by Mudholkar and Srivastava (1993). This method of generalizing a baseline distribution has been shown by several authors to provide better fit and more flexibility than its baseline distribution. Pal et al. (2006) studied the Exponentiated Weibull distribution introduced by Mudholkar and Srivastava (1993) and compared it with the two-parameter Weibull and gamma distributions using real life data. The result showed that the Exponentiated Weibull distribution gave better fit. Nadarajah and Kotz (2006) introduced the Exponentiated Fréchet distribution, a generalization of the standard Fréchet distribution. Flaih et al. (2012) introduced the Exponentiated Inverted Weibull distribution which works better than the Inverted Weibull distribution. Gupta and Kundu (1999) proposed the generalized exponential distributions which they showed to be better than Weibull and Gamma distributions, respectively.

In this study, we propose a new distribution known as the Exponentiated Akash distribution which is a generalization of the Akash distribution. The characteristics of the new distribution were also derived. The rest of the paper is organized as follows: Section 2 contains the mathematical characteristics of the new distribution, Section 3 contains the maximum likelihood estimates, Section 4 contains the application and comparison of the new distribution with some lifetime distributions using two real life data sets and finally, Section 5 is the conclusion of the paper.

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The probability density function (pdf) and cumulative density function (cdf) of the Akash distribution as proposed by Shanker (2015) is given as;

$$f(x) = \frac{\theta^3}{\theta^2+2}(1+x^2)e^{-\theta x}, x > 0, \theta > 0 \quad (1)$$

$$F(x) = 1 - \left[1 + \frac{\theta x(\theta x+2)}{\theta^2+2}\right] e^{-\theta x};\ x > 0, \theta > 0 \quad (2)$$

A random variable X is said to have an exponentiated distribution if its pdf and cdf are given, respectively, by;

$$D_\alpha(x) = [F(x)]^\alpha; Z \in R', \alpha > 0 \quad (3)$$

$$d_\alpha(x) = \alpha[F(x)]^{\alpha-1} f(x) \quad (4)$$

Hence, we obtain the pdf and cdf of the new Exponentiated Akash distribution using the equations above as;

$$D_\alpha(x) = \left[1 - \left[1 + \frac{\theta x(\theta x+2)}{\theta^2+2}\right] e^{-\theta x}\right]^\alpha \quad (5)$$

$$d_\alpha(x) = \alpha \frac{\theta^3}{\theta^2+2}(1+x^2)\left[1 - \left[1 + \frac{\theta x(\theta x+2)}{\theta^2+2}\right] e^{-\theta x}\right]^{\alpha-1} e^{-\theta x} \quad (6)$$

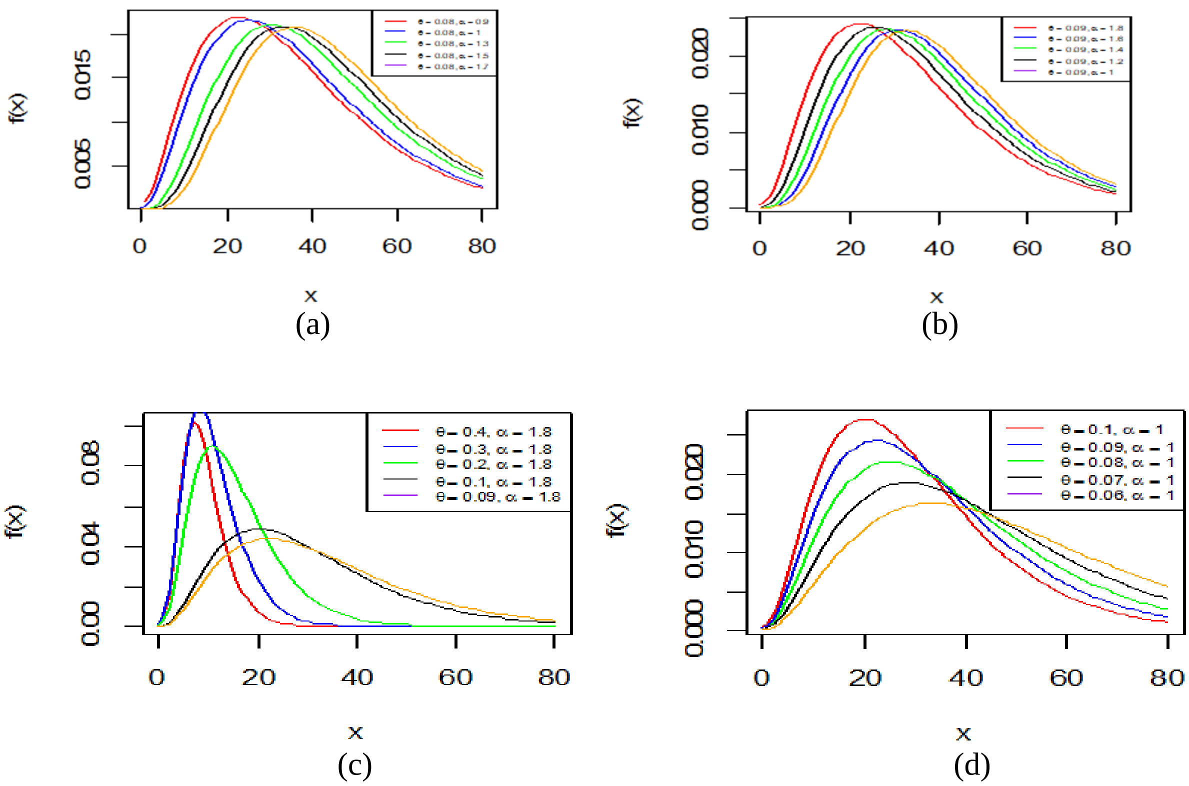


Figure 1: (a), (b), (c) and (d) Showing the PDF of the Exponentiated Akash Distribution

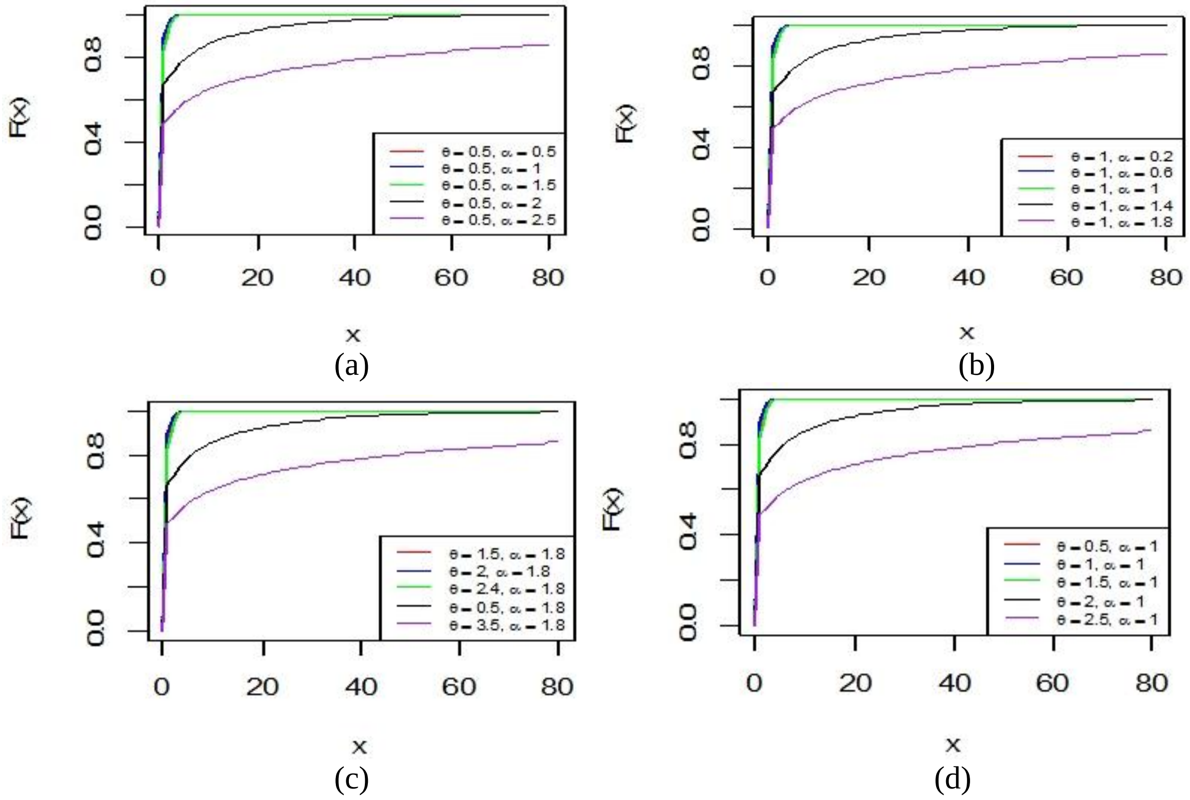


Figure 2: (a), (b), (c) and (d) Showing the Shapes of the CDF of the Exponentiated Akash Distribution

2. Mathematical Characteristics

2.1 Moments

Theorem 1

Given a random variable X, the r^{th} moments about origin, $E(X^r)$, of the Exponentiated Akash distribution is given by

$$E(X^r) = A_{i,j,k}\left[\frac{(r+2j-k)!}{(i+1)^{r+2j-k+1}}\right] + B_{i,j,k}\left[\frac{(r+2j-k+2)!}{(i+1)^{r+2j-k+3}}\right], \tag{7}$$

where $A_{i,j,k} = \sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i \sum_{j=0}^{i}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\frac{\alpha\cdot 2^k\cdot\theta^{-r+2}}{(\theta^2+2)^{j+1}}$

$and\ B_{i,j,k} = \sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i \sum_{j=0}^{i}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\frac{\alpha\cdot 2^k\cdot\theta^{-r}}{(\theta^2+2)^{j+1}}.$

Proof

The moment of a random variable, X, is given by

$$E(X^r) = \int_0^\infty x^r d_\alpha(x)dx$$

$$= \int_0^\infty \frac{\alpha\theta^3 x^r}{\theta^2+2}(1+x^2)\left[1-\left[1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right]e^{-\theta x}\right]^{\alpha-1} e^{-\theta x}dx$$

$$= \int_0^\infty \frac{\alpha\theta^3 x^r}{\theta^2+2} e^{-\theta x}\left[1-\left[1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right]e^{-\theta x}\right]^{\alpha-1} dx$$

$$+ \int_0^\infty \frac{\alpha\theta^3 x^{r+2}}{\theta^2+2} e^{-\theta x}\left[1-\left[1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right]e^{-\theta x}\right]^{\alpha-1} dx$$

Using binomial expansion,

$$\left[1-\left[1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right]e^{-\theta x}\right]^{\alpha-1} = \sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i\left[1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right]^i e^{-i\theta x}$$

$$= \sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i \sum_{j=0}^{i}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\frac{2^k(\theta x)^{2j-k}}{(\theta^2+2)^j} e^{-i\theta x}$$

Substituting, we have

$$E(X^r) = \sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i \sum_{j=0}^{i}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\frac{\alpha\cdot 2^k\cdot\theta^{2j-k+3}}{(\theta^2+2)^{j+1}}\int_0^\infty x^{r+2j-k}\cdot e^{-\theta x(i+1)}dx$$

$$+ \sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i \sum_{j=0}^{i}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\frac{\alpha\cdot 2^k\cdot\theta^{2j-k+3}}{(\theta^2+2)^{j+1}}\int_0^\infty x^{r+2j-k+2}\cdot e^{-\theta x(i+1)}dx$$

Recall $\int_0^\infty x^n e^{-ax}dx = \frac{\Gamma(n+1)}{a^{n+1}}$ and $\Gamma(\alpha) = (\alpha-1)!$

Therefore,

$$E(X^r) = \sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i \sum_{j=0}^{i}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\frac{\alpha 2^k\theta^{-r+2}}{(\theta^2+2)^{j+1}}\frac{(r+2j-k)!}{(i+1)^{r+2j-k+1}}$$

$$+ \sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i \sum_{j=0}^{i}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\frac{\alpha 2^k\theta^{-r}}{(\theta^2+2)^{j+1}}\frac{(r+2j-k+2)!}{(i+1)^{r+2j-k+3}}$$

Let

$A_{i,j,k} = \sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i \sum_{j=0}^{i}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\frac{\alpha\cdot 2^k\cdot\theta^{-r+2}}{(\theta^2+2)^{j+1}}$ *and* $B_{i,j,k} =$

$\sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i \sum_{j=0}^{i}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\frac{\alpha\cdot 2^k\cdot\theta^{-r}}{(\theta^2+2)^{j+1}}$.

Therefore,

$$E(X^r) = A_{i,j,k}\left[\frac{(r+2j-k)!}{(i+1)^{r+2j-k+1}}\right] + B_{i,j,k}\left[\frac{(r+2j-k+2)!}{(i+1)^{r+2j-k+3}}\right].$$

2.2 Moment Generating Function

Theorem 2

Suppose X follows an Exponentiated Akash distribution. Then the moment generating function of X, $M_X(t)$, is given by

$$M_X(t) = \sum_{l=0}^{\infty}\left(\frac{t}{\theta}\right)^l\left[A_{i,j,k}\cdot\frac{(2j-k+l)!}{l!(i+1)^{r+2j-k+1}} + B_{i,j,k}\cdot\frac{(2j-k+l+2)!}{l!(i+1)^{r+2j-k+1}}\right] \quad (8)$$

where $A_{i,j,k} = \sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i\sum_{j=0}^{i}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\frac{\alpha\cdot 2^k\cdot\theta^{-2}}{(\theta^2+2)^{j+1}}$

and $B_{i,j,k} = \sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i\sum_{j=0}^{i}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\frac{\alpha\cdot 2^k}{(\theta^2+2)^{j+1}}$

Proof

The moment generating function of a random variable, X, is given by

$$M_x(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} d_{\alpha}(x)dx$$

$$= \int_0^{\infty} e^{tx}\frac{\alpha\theta^3}{\theta^2+2}(1+x^2)\left[1-\left[1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right]e^{-\theta x}\right]^{\alpha-1} e^{-\theta x}dx$$

$$= \int_0^{\infty}\frac{\alpha\theta^3}{\theta^2+2}e^{-x(\theta-t)}\left[1-\left[1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right]e^{-\theta x}\right]^{\alpha-1}dx + \int_0^{\infty}\frac{\alpha\theta^3x^2}{\theta^2+2}e^{-x(\theta-t)}\left[1-\left[1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right]e^{-\theta x}\right]^{\alpha-1}dx$$

Using binomial expansion,

$$\left[1-\left[1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right]e^{-\theta x}\right]^{\alpha-1} = \sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i\left[1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right]^i e^{-i\theta x}$$

$$= \sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i\sum_{j=0}^{i}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\frac{2^k(\theta x)^{2j-k}}{(\theta^2+2)^j}e^{-i\theta x}$$

$Also, e^{tx} = \sum_{l=0}^{\infty}\frac{(tx)^l}{l!}$. Substituting, we have

$$M_X(t) = \sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i\sum_{j=0}^{i}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{\infty}\frac{(t)^l}{l!}\frac{\alpha 2^k\theta^{2j-k+3}}{(\theta^2+2)^{j+1}}\int_0^{\infty}x^{2j-k+l}e^{-\theta x(i+1)}dx + \sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i\sum_{j=0}^{i}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{\infty}\frac{(t)^l}{l!}\frac{\alpha 2^k\theta^{2j-k+3}}{(\theta^2+2)^{j+1}}\int_0^{\infty}x^{l+2j-k+2}e^{-\theta x(i+1)}dx$$

Recall $\int_0^\infty x^n e^{-ax} dx = \frac{\Gamma(n+1)}{a^{n+1}}$ and $\Gamma(\alpha) = (\alpha - 1)!$

Therefore,

$$M_X(t) = \sum_{i=0}^{\infty} \binom{\alpha-1}{i} (-1)^i \sum_{j=0}^{i} \binom{i}{j} \sum_{k=0}^{j} \binom{j}{k} \sum_{l=0}^{\infty} \left(\frac{t}{\theta}\right)^l \frac{\alpha \cdot 2^k \cdot \theta^{-2}}{(\theta^2+2)^{j+1}} \cdot \frac{(2j-k+l)!}{l!\,(i+1)^{2j-k+l+1}}$$

$$+ \sum_{i=0}^{\infty} \binom{\alpha-1}{i} (-1)^i \sum_{j=0}^{i} \binom{i}{j} \sum_{k=0}^{j} \binom{j}{k} \sum_{l=0}^{\infty} \left(\frac{t}{\theta}\right)^l \frac{\alpha \cdot 2^k}{(\theta^2+2)^{j+1}} \cdot \frac{(2j-k+l+2)!}{(i+1)^{2j-k+l+3}}$$

Let

$A_{i,j,k} = \sum_{i=0}^{\infty} \binom{\alpha-1}{i} (-1)^i \sum_{j=0}^{i} \binom{i}{j} \sum_{k=0}^{j} \binom{j}{k} \frac{\alpha \cdot 2^k \cdot \theta^{-2}}{(\theta^2+2)^{j+1}}$ *and* $B_{i,j,k} =$

$\sum_{i=0}^{\infty} \binom{\alpha-1}{i} (-1)^i \sum_{j=0}^{i} \binom{i}{j} \sum_{k=0}^{j} \binom{j}{k} \frac{\alpha \cdot 2^k}{(\theta^2+2)^{j+1}}$.

Therefore,

$$M_X(t) = \sum_{l=0}^{\infty} \left(\frac{t}{\theta}\right)^l \left[A_{i,j,k} \cdot \frac{(2j-k+l)!}{l!\,(i+1)^{r+2j-k+1}} + B_{i,j,k} \cdot \frac{(2j-k+l+2)!}{l!\,(i+1)^{r+2j-k+1}}\right].$$

2.3 Order Statistics

Theorem 3

Suppose $X_1, X_2, \ldots, X_n$ is a random sample from an exponentiated Akash distribution. Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ denote the corresponding order statistics. Then, the probability density function and the cumulative distribution function of the p^{th} order statistics, say, $Y = X_{(p)}$, is given, respectively, by

$$f_X(x) = \sum_{i=0}^{n-p} \binom{n-p}{i} (-1)^i \sum_{j=0}^{\infty} \binom{\alpha(p+i)-1}{j} (-1)^j \sum_{k=0}^{j} \binom{j}{k} \sum_{l=0}^{k} \binom{k}{l} \frac{\alpha \cdot n! \cdot \theta^{2k-l+2} \cdot (1+x^2) \cdot x^{2k-l} \cdot e^{-\theta x(j+1)}}{(\theta^2+2)^{k+1} (p-1)!(n-p)!} \quad (9)$$

$$F_X(x) = \sum_{i=p}^{n} \sum_{j=0}^{n-l} \sum_{k=0}^{\infty} \sum_{l=0}^{k} \sum_{m=0}^{l} \binom{n}{i} \binom{n-i}{j} (-1)^j \binom{\alpha(i+j)}{k} (-1)^k \binom{k}{l} \binom{l}{m} \frac{2^m \cdot (\theta x)^{2l-m} \cdot e^{-k\theta x}}{(\theta^2+2)^l} \quad (10)$$

Proof

The pdf of the p^{th} order statistics is given by

$$f_X(x) = \frac{n!}{(p-1)!\,(n-p)!} {D_\alpha}^{p-1}(x)[1 - d_\alpha(x)]^{n-p}$$

$$f_X(x) = \frac{n!}{(p-1)!(n-p)!} \sum_{i=0}^{n-p} \binom{n-p}{i} (-1)^i {D_\alpha}^{p-1+i}(x) d_\alpha(x) \quad (11)$$

Substituting for $D_\alpha(x)$ and $d_\alpha(x)$ in (11) above we obtain

$$f_X(x) = \frac{\alpha\theta^3 n!\,(1+x^2)}{(\theta^2+2)(p-1)!\,(n-p)!}\sum_{i=0}^{n-p}\binom{n-p}{i}(-1)^i\left[1-\left[1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right]e^{-\theta x}\right]^{\alpha-1} e^{-\theta x}\left[1 - \left[1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right]e^{-\theta x}\right]^{\alpha(p-1+i)}$$

$$f_X(x) = \frac{\alpha\theta^3 n!\,(1+x^2)e^{-\theta x}}{(\theta^2+2)(p-1)!\,(n-p)!}\sum_{i=0}^{n-p}\binom{n-p}{i}(-1)^i\left[1-\left[1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right]e^{-\theta x}\right]^{\alpha(p+i)-1}$$

Using binomial expansion,

$$\left[1-\left[1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right]e^{-\theta x}\right]^{\alpha(p+i)-1} = \sum_{j=0}^{\infty}\binom{\alpha(p+i)-1}{j}(-1)^j\left[1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right]^j e^{-j\theta x}$$

$$= \sum_{j=0}^{\infty}\binom{\alpha(p+i)-1}{j}(-1)^j\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}\frac{2^l\cdot(\theta x)^{2k-l}}{(\theta^2+2)^k}e^{-j\theta x}.$$

Therefore,

$$f_X(x) = \sum_{i=0}^{n-p}\binom{n-p}{i}(-1)^i\sum_{j=0}^{\infty}\binom{\alpha(p+i)-1}{j}(-1)^j\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}\frac{\alpha n!\theta^{2k-l+2}(1+x^2)x^{2k-l}e^{-\theta x(j+1)}}{(\theta^2+2)^{k+1}(p-1)!(n-p)!}.$$

The cdf of the p^{th} order statistics is given by

$$F_X(x) = \sum_{i=p}^{n}\binom{n}{i}D_\alpha{}^i(x)\,[1-D_\alpha(x)]^{n-i}$$

$$F_X(x) = \sum_{i=p}^{n}\sum_{j=0}^{n-i}\binom{n}{i}\binom{n-i}{i}(-1)^j D_\alpha{}^{i+j}(x) \qquad (12)$$

Substituting, we have

$$F_X(x) = \sum_{i=p}^{n}\sum_{j=0}^{n-i}\binom{n}{i}\binom{n-i}{j}(-1)^j\left[1-\left[1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right]e^{-\theta x}\right]^{\alpha(i+j)}.$$

Using binomial expansion,

$$\left[1-\left[1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right]e^{-\theta x}\right]^{\alpha(i+j)} = \sum_{k=0}^{\infty}\binom{\alpha(i+j)}{k}(-1)^k\left[1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right]^k e^{-k\theta x}$$

$$= \sum_{k=0}^{\infty}\binom{\alpha(i+j)}{k}(-1)^k\sum_{l=0}^{k}\binom{k}{l}\sum_{m=0}^{l}\binom{l}{m}\frac{2^m\cdot(\theta x)^{2l-m}}{(\theta^2+2)^l}e^{-k\theta x}.$$

Hence,

$$F_X(x) = \sum_{i=p}^{n}\sum_{j=0}^{n-l}\sum_{k=0}^{\infty}\sum_{l=0}^{k}\sum_{m=0}^{l}\binom{n}{i}\binom{n-i}{j}(-1)^j\binom{\alpha(i+j)}{k}(-1)^k\binom{k}{l}\binom{l}{m}\frac{2^m\cdot(\theta x)^{2l-m}\cdot e^{-k\theta x}}{(\theta^2+2)^l}.$$

2.4 Entropy

Entropy is one of the major properties of a probability distribution. It is basically used to measure the uncertainties obtained in a probability distribution. Different forms of entropy have been proposed such as the Rényi entropy, the Shannon entropy, etc. Here, we shall consider the widely used Rényi entropy.

Theorem 4

Given a random variable, X, which follows an Exponentiated Akash distribution, the Rényi entropy is given by

$$T_R(\beta) = \frac{1}{1-\beta} log\left\{\sum_{i=0}^{\infty}\binom{\beta}{i}\sum_{j=0}^{\infty}\binom{\beta(\alpha-1)-1}{j}(-1)^j\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}\frac{2\cdot\alpha^{\beta}\cdot\theta^{3\beta-2i}}{(\theta^2+2)^{k+\beta}}\frac{(2i+2k-l)!}{\beta(j+1)^{2i+2k-l+1}}\right\} \quad (13)$$

Proof.

The Rényi entropy is given by

$$T_R(\beta) = \frac{1}{1-\beta} log\left[\int d_{\alpha}{}^{\beta}(x)dx\right];\ \beta > 0\ and\ \beta \neq 1.$$

$$T_R(\beta) = \frac{1}{1-\beta} log\left\{\int_0^{\infty}\left[\frac{\alpha\theta^3}{\theta^2+2}(1+x^2)\left[1-\left[1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right]e^{-\theta x}\right]^{\alpha-1}e^{-\theta x}\right]^{\beta}dx\right\}.$$

$$T_R(\beta) = \frac{1}{1-\beta} log\left\{\int_0^{\infty}\frac{\alpha^{\beta}\cdot\theta^{3\beta}}{(\theta^2+2)^{\beta}}(1+x^2)^{\beta}\left[1-\left[1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right]e^{-\theta x}\right]^{\beta(\alpha-1)}e^{-\theta\beta x}dx\right\}.$$

Using the binomial expansion,

$$\left[1-\left[1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right]e^{-\theta x}\right]^{\beta(\alpha-1)} = \sum_{j=0}^{\infty}\binom{\beta(\alpha-1)}{j}(-1)^j\left[1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right]^j e^{-j\theta x}$$

$$= \sum_{j=0}^{\infty}\binom{\beta(\alpha-1)}{j}(-1)^j\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}\frac{2^l\cdot(\theta x)^{2k-l}}{(\theta^2+2)^k}e^{-j\theta x}.$$

Substituting, we have

$$T_R(\beta) = \frac{1}{1-\beta} log\left\{\sum_{j=0}^{\infty}\binom{\beta(\alpha-1)}{j}(-1)^j\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}\frac{2^l\cdot\alpha^{\beta}\cdot\theta^{3\beta+2k-l}}{(\theta^2+2)^k}\int_0^{\infty}x^{2i+2k-l}e^{-\beta\theta x(j+1))}dx\right\}.$$

Recall, $\int_0^{\infty}x^n e^{-ax}dx = \frac{\Gamma(n+1)}{a^{n+1}}$ and $\Gamma(\alpha) = (\alpha-1)!$

Therefore,

$$T_R(\beta) = \frac{1}{1-\beta} log\left\{\sum_{i=0}^{\infty}\binom{\beta}{i}\sum_{j=0}^{\infty}\binom{\beta(\alpha-1)-1}{j}(-1)^j\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}\frac{2\cdot\alpha^{\beta}\cdot\theta^{3\beta-2i}}{(\theta^2+2)^{k+\beta}}\cdot\frac{(2i+2k-l)!}{\beta(j+1)^{2i+2k-l+1}}\right\}.$$

2.5 Reliability Analysis

Given any probability distribution, the reliability analysis is always considered based on the survival function and the hazard rate function of the distribution. Hence, for the Exponentiated Akash distribution, the survival and hazard rate function are given below.

2.5.1 Survival function

The probability that an item does not fail prior to some time, t is known as the survival function. This function is considered one of the important functions in reliability analysis. It is given by

$$S_\alpha(x) = 1 - D_\alpha(x) \qquad (14)$$

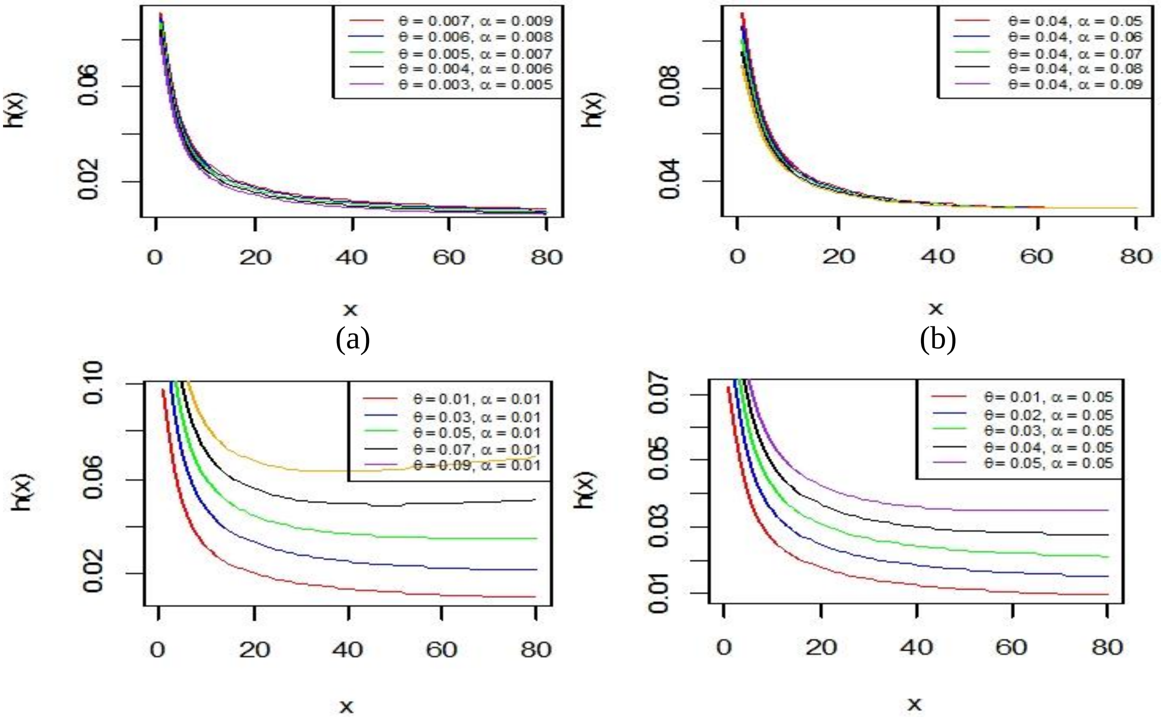
$$S_\alpha(x) = 1 - \left(\left[1 - \left[1 + \frac{\theta x(\theta x + 2)}{\theta^2 + 2}\right] e^{-\theta x}\right]\right)^\alpha$$

2.5.2 Hazard rate function

The hazard rate function is another important function in reliability analysis. It is defined as the conditional probability of failure of an item given it has survived to the time, t. It is given by

$$H_\alpha(x) = \frac{d_\alpha(x)}{1 - D_\alpha(x)} \qquad (16)$$

$$h_\alpha(x) = \frac{\frac{\alpha\theta^3}{\theta^2+2}(1+x^2)\left[1-\left[1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right]e^{-\theta x}\right]^{\alpha-1} e^{-\theta x}}{1-\left(\left[1-\left[1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right]e^{-\theta x}\right]\right)^\alpha} \qquad (17)$$



(a) (b) (c) (d)

Figure 3: The Hazard rate Function of the Exponentiated Akash Distribution

The plots of the hazard rate function above shows that it is a monotone decreasing function.

3. Maximum Likelihood Estimation

Let $X_1, X_2, \ldots, X_n$ be a random sample of size, n, from the Exponentiated Akash distribution, the log-likelihood function of the parameters can be written as

$$LL(\alpha, \theta) = \prod_{i=1}^{n} \ln d_\alpha(x_i) \qquad (18)$$

$$= \prod_{i=1}^{n} \ln\left\{\frac{\alpha\theta^3}{\theta^2+2}(1+x_i^2)\left[1-\left[1+\frac{\theta x_i(\theta x_i+2)}{\theta^2+2}\right]e^{-\theta x_i}\right]^{\alpha-1} e^{-\theta x_i}\right\}$$

$$= \ln\left\{\left(\frac{\alpha\theta^3}{\theta^2+2}\right)^n \prod_{i=1}^{n}\left(1+x_i^2\right) e^{-\theta\sum x_i} \prod_{i=1}^{n}\left[1-\left[1+\frac{\theta x_i(\theta x_i+2)}{\theta^2+2}\right] e^{-\theta x_i}\right]^{\alpha-1}\right\}.$$

$$LL = n[\ln(\alpha) + 3\ln(\theta) - \ln(\theta^2+2)] + \sum_{i=1}^{n}\ln\left(1+x_i^2\right) - \theta\sum_{i=1}^{n} x_i + (\alpha-1)\sum_{i=1}^{n}\ln\left(A_i(\theta)\right) \quad (19)$$

where $A_i(\theta) = \left[1-\left[1+\frac{\theta x_i(\theta x_i+2)}{\theta^2+2}\right] e^{-\theta x_i}\right]$.

In order to maximize the log likelihood, we solve the nonlinear likelihood equations simultaneously obtained from the differentiation of (19) with respect to θ and α as shown below.

$$\frac{\partial LL}{\partial\theta} = \partial\left\{n[\ln(\alpha) + 3\ln(\theta) - \ln(\theta^2+2)] - \theta\sum_{i=1}^{n} x_i + (\alpha-1)\sum_{i=1}^{n}\ln\left(A_i(\theta)\right)\right\}.$$

$$\frac{\partial LL}{\partial\theta} = \frac{n[3(\theta^2+2) - 2\theta^2]}{\theta(\theta^2+2)} - \sum_{i=1}^{n} x_i + \frac{(\alpha-1)\left[\frac{2\theta^2 x_i\left(2x_i^2 - 1 - \theta x_i\right) + 4\theta x_i^2}{(\theta^2+2)^2}\right]}{1-\left[1+\frac{\theta x_i(\theta x_i+2)}{\theta^2+2}\right] e^{-\theta x_i}}. \quad (20)$$

$$\frac{\partial LL}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\ln\left(A_i(\theta)\right). \quad (21)$$

The parameter estimates above are obtained using the maxLik package in the R software.

4. Applications

In order to illustrate the usefulness of the Exponentiated Akash distribution, we present in this section, the goodness of fit of the Exponentiated Akash distribution on two real life data-sets and compare it with some lifetime distributions such as the Exponential, Lindley distribution (LD) by Lindley (1958), Akash distribution (AD) by Shanker (2015), Two-parameter Akash (TPAD) by Shanker and Shukla (2017), Exponentiated Lindley distribution (ELD) by Bakouch et al. (2012) and Exponentiated Exponential distributions (EED) by Gupta and Kundu (1999).

The measures adopted for the goodness of fit testing are the parameter estimates, the Akaike Information Criteria (AIC), Bayesian Information Criteria (BIC) and the Kolmogorov-Smirnov Statistic (K-S).

The real life datasets used are presented in the tables below.

Dataset 1: This data is related with behavioural sciences, collected by Balakrishnan et al. (2010). The scale "General Rating of Affective Symptoms for preschoolers (GRASP)" measures behavioural and emotional problems of children, which can be classified with depressive condition or not according to this scale. A study by the authors in a city located at the southern part of Chile has allowed collecting real data corresponding to the scores of the GRASP scale of children with frequency in parenthesis, as shown below.

19(16)	20(15)	21(14)	22(9)	23(120	24(10)
25(6)	26(9)	27(8)	28(5)	29(6)	30(4)
31(3)	32(4)	33	34	35(4)	36(2)
37(2)	39	42	44		

Dataset 2: This data contains the strength of data of glass of the aircraft window as reported by Fuller et al. (1994) and shown below.

18.83	20.8	21.657	23.03	23.23	24.05
24.321	25.5	25.52	25.8	26.69	26.77
26.78	27.05	27.67	29.9	31.11	33.2
33.73	33.76	33.89	34.76	35.75	35.91
36.98	37.08	37.09	39.58	44.045	45.29
45.381					

From Table 3 above, it can be easily seen that the Exponentiated Akash distribution gave better fit than the other competing distributions.

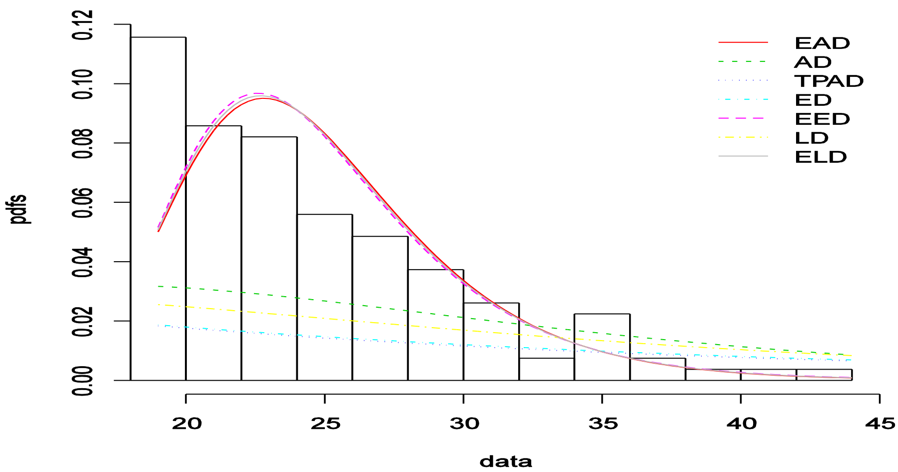


Figure 4: The fitted pdf functions of some selected lifetime distribution and the exponentiated Akash distribution

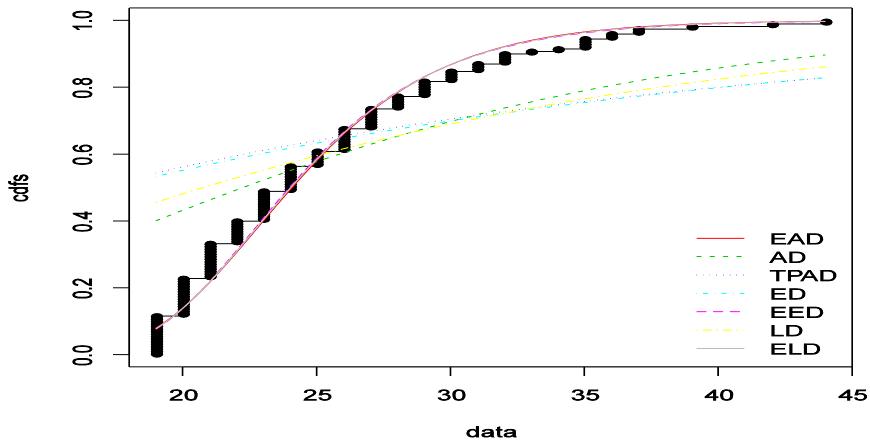


Figure 5: The fitted cdf functions of some selected lifetime distribution and the exponentiated Akash distribution

Table 3: MLE's, -2ln L, AIC, BIC, K-S Statistics of the fitted distributions of datasets

Data	Model	Parameter Estimates	-2LL	AIC	BIC	K-S
Data1	EAD	$\boldsymbol{\hat{\theta} = 0.0249}$ $\boldsymbol{\hat{\alpha} = 0.6837}$	**789.8818**	**793.8819**	**798.4352**	**0.1491**
	AD	0.1196	981.2844	983.2843	986.1822	0.4002
	TPAD	$\hat{\theta} = 0.4528 \times 10^{-2}$ $\hat{\lambda} = 1.3550 \times 10^{4}$	1138.532	1142.453	1148.249	0.5434
	ED	0.0100	806.8842	808.8842	811.1609	0.2116
	EED	$\hat{\theta} = 0.2627$ $\hat{\alpha} = 378.6226$	792.148	796.1479	801.9436	0.1179
	LD	0.0772	1041.6442	1043.644	1046.542	0.4556
	ELD	$\hat{\theta} = 0.2968$ $\hat{\beta} = 133.3423$	793.4308	797.4307	803.2264	0.1191
Data 2	**EAD**	$\boldsymbol{\hat{\theta} = 0.2212}$ $\boldsymbol{\hat{\alpha} = 17.0929}$	**208.16**	**212.16**	**215.028**	**0.1317**
	AD	0.0971	240.6818	242.6818	244.1157	0.2987
	TPAD	$\hat{\theta} = 3.861 \times 10^{-2}$ $\hat{\lambda} = 1.280 \times 10^{4}$	274.53	278.5301	281.398	0.4584
	ED	0.0325	274.5288	276.5289	277.9629	0.4586
	EED	$\hat{\theta} = 0.1660$ $\hat{\alpha} = 33.7818$	208.2686	212.6286	215.1366	0.1360
	LD	0.0630	253.9884	255.9884	257.4224	0.3655
	ELD	$\hat{\theta} = 0.1934$ $\hat{\beta} = 36.6683$	208.2038	212.2038	215.0718	0.1337

Figures 4 and 5 show the pdf and the cdf of the Exponentiated Akash distribution and the other competing distributions of Data 2. From the fitted plots, the Exponentiated Akash distribution gave a better fit to the data than the other competing distributions.

5. Conclusion

In conclusion, a new distribution which is a generalization of the Akash distribution known as the Exponentiated Akash distribution was introduced. The characteristics of this distribution such as the Moments and Moment generating function, order statistics, entropy and reliability analysis was discussed. The maximum likelihood estimation of the Exponentiated Akash distribution was also presented with its application using two real life data sets. From the results presented, the Exponentiated Akash distribution performed well, giving better fits than some of the lifetime distributions it was compared with.

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