Special classes of multivariate generalized autoregressive conditional heteroscedasticity models for volatility series

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In this paper, we focused on Multivariate Generalised Autoregressive Conditional Heteroscedasticity models for volatility series using response vector of variances. The paper aimed at developing alternative multivariate GARCH models characterised by either autoregressive or moving average process. Isolated Multivariate Generalised Conditional Heteroscedasticity, ISO-MGARCH (p,0) models and Isolated Multivariate Generalised Conditional Heteroscedasticity, ISO-MGARCH(0, q) models are identified from MGARCH (p, q) model under specific conditions. To ascertain the models applicability, the isolated univariate and multivariate GARCH (2,0) models were fitted to volatility measures of Nigeria average, urban and rural consumer price indices from January 1995 to December 2019. The volatility series were subjected to autocorrelation and partial autocorrelation checks as applicable to stationary autoregressive moving average process, where single autoregressive and moving average models are identified under certain conditions. This justified the isolation of pure autoregressive and pure moving average MGARCH models. Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC) and Schwarz's Information criterion (SIC) compare the isolated multivariate GARCH models with the existing univariate GARCH models, and the results revealed the same comparative advantage in capturing volatility series.

Keywords: MGARCH(p, q); ISO-MGARC(p, 0); ISO-MGARCH(0, q); volatility measures.

1 Introduction

Autoregressive Conditional Heteroscedasticity (ARCH) and Generalised Autoregressive Conditional Heteroscedasticity (GARCH) models have gained priority in the modelling of economic and financial time series. These models have been established and found useful in investigating volatility in dynamic series, Engle (1982). Multivariate ARCH and GARCH models are extension of the univariate ARCH and GARCH models. Bollerslev et al (1988) and Hansson and Hordahl (1998) developed Multivariate GARCH models and used it to investigate volatilities of asset in portfolio, risk management and asset allocation to finding and updating optimal hedging positions. In multivariate ARCH and GARCH models, each conditional variance or covariance is a function of its lagged error term, variance and that of the other predictor conditional variances and error components. Silvennoinen and Terӓsvirta (2009) proposed specifications of MGARCH models which included flexibility to represent the dynamics of the conditional variances and covariances. Given the increasing number of parameters in MARCH and MGARCH models, Silvennoinen and Terӓsvirta (2009) considered parsimonious model for relative easy estimation of parameters. Although parsimonious model is synonymous with few parameters, this kind of model may not be able to capture the relevant dynamics in the covariance structure. Also taken into consideration is

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the feature of the positive definiteness of the covariance matrices. In this paper, we consider the response vector of variances a linear combinations of the lagged terms of the response and predictor variances and associated squared errors. The objectives of this work are to obtain the multivariate generalized conditional heteroscedasticity (MGARCH) model for volatility series and also identify alternative MGARCH models under certain conditions as applicable to autoregressive moving average models.

2 Review of Models

This section considers the review of the related multivariate autoregressive conditional heteroscedasticity model and generalised multivariate autoregressive conditional heteroscedasticity models.

Given $U_t = (U_{1t}, U_{2t}, \dots, U_{kt})^I$ dimensional zero mean, serially uncorrelated process which may be the residual process of some dynamic model, represented in the form

$$U_t = \sum_{t|t-1}^{1/2} \epsilon_t \tag{1}$$

where, ϵ_t is the k-dimensional iid white noise, where $\epsilon_t \sim iid(0, I_k)$ and $\sum_{t|t-1}$is the conditional covariance matrix of U_t, given $U_{t-1}, U_{t-2,}$ as usual $\sum_{t|t-1}^{1/2}$ is the symmetric positive definite square root of $\sum_{t|t-1}$. The representation of Multivariate ARCH (q) process is

$$Vech(\textstyle\sum_{t|t-1}) = \gamma_0 + \Gamma_I Vech(U_{t-1}U_{t-1}^I) + \dots + \Gamma_q Vech\left(U_{t-q}U_{t-q}^I\right) \tag{2}$$

where, *Vech* denotes the half-vectorization operator which stacks the columns of a square matrix from the diagonal downwards in a vector, γ_0 is a $\frac{1}{2}K(K+1)$ dimensional vector of constants and the $\Gamma_j's$ are $\left[\frac{1}{2}k(k+1) \times \frac{1}{2}k(k+1)\right]$ coefficients matrices. Borlerslev et al (1988) considered multivariate ARCH models of the form k=2 ARCH(1) process as

$$Vech\begin{bmatrix}\sigma_{11,t|t-1} & \sigma_{12,t|t-1}\\ \sigma_{21,t|t-1} & \sigma_{22,t|t-1}\end{bmatrix} = \begin{bmatrix}\sigma_{11,t|t-1}\\ \sigma_{12,t|t-1}\\ \sigma_{22,t|t-1}\end{bmatrix} = \begin{bmatrix}\gamma_{10}\\ \gamma_{20}\\ \gamma_{30}\end{bmatrix} + \begin{bmatrix}\gamma_{11} & \gamma_{12} & \gamma_{13}\\ \gamma_{21} & \gamma_{22} & \gamma_{23}\\ \gamma_{31} & \gamma_{32} & \gamma_{33}\end{bmatrix}\begin{bmatrix}U_{1,t-1}^2\\ U_{1,t-1}U_{2,t-1}\\ U_{2,t-1}^2\end{bmatrix} \tag{3}$$

Equation 3 is a multivariate ARCH model for volatility series. In order to reduce the number of parameters in the above multivariate ARCH model for k=2 ARCH(1) model, Borllerslev et al. (1988) presented a more parsimonious model by considering multivariate diagonal ARCH process, where the parameter matrices are all pure diagonal. This is obtained as

$$\begin{bmatrix}\sigma_{11,t|t-1}\\ \sigma_{12,t|t-1}\\ \sigma_{22,t|t-1}\end{bmatrix} = \begin{bmatrix}\gamma_{10}\\ \gamma_{20}\\ \gamma_{30}\end{bmatrix} + \begin{bmatrix}\gamma_{11} & 0 & 0\\ 0 & \gamma_{22} & 0\\ 0 & 0 & \gamma_{33}\end{bmatrix}\begin{bmatrix}U_{1,t-1}^2\\ U_{1,t-1}U_{2,t-1}\\ U_{2,t-1}^2\end{bmatrix} \tag{4}$$

Equation (4) is a multivariate pure diagonal ARCH model, also called diagonal vector (DVEC) model, which reduces the number of parameters in the ARCH (1) model. Silvennoinen and Teräsvirta (2009) defined a stochastic vector process (r_t) with dimension kx1 such that $E(r_t) = 0$. $r_t$is conditional heteroskedastic,

$$r_t = H_t^{1/2} n_t \tag{5}$$

where H_t is $k \times k$ conditional covariance matrix of r_t and n_t is $k \times 1$ iid vector error process such that $E(n_t n_t^I) = 0$. This defines the standard multivariate GARCH framework, in which there is no linear dependence structure in r_t. The model precludes parametric formulation of the linear dependence of H_t. Bollerslev et al (1988) presented a straightforward Vector GARCH model as a generalisation of the univariate GARCH model. The model is written, thus

$$Vech(H_t) = C + \sum_{j=1}^{q} A_j Vech(r_{t-j} r_{t-j}^I) + \sum_{j=1}^{p} B_j Vech(H_{t-j}) \quad (6)$$

where, $H_t$is the variance and covariance matrix with $k \times k$ dimension, C is an $\frac{K(K+1)}{2}$ $Vector, and\ A_j\ and\ B_j\ are$ $\left[\frac{1}{2}k(k+1) \times \frac{1}{2}k(k+1)\right]$ parameter matrices. Similar to multivariate ARCH(1) model in equation (4), Bollerslev (1988) introduced Diagonal Vectorization (DVEC) model, where the parameter matrices $A_j\ and\ B_j$ contain only principal diagonal elements, Luc et al (2006). The DVEC model has parsimonious advantage, since the off-diagonal coefficients are zeroed. A major drawback is that the response vector of variances is only dependent on its distributed lags and does not allow for a greater range of interactions with other lagged variances. The restricted version of the VEC model is the Baba-Engle-Kraft-Kroner (BEKK) defined in Engle and Kroner (1995). The conditional covariance matrices are positive definite by construction. The model has the form

$$H_t = CC^I + \sum_{j=1}^{q}\sum_{k=1}^{k} A_{kj}^I r_{t-j} A_{kj} + \sum_{j=1}^{p}\sum_{k=1}^{k} B_{kj}^I H_{t-j} B_{kj} \quad (7)$$

where, $A_{kj}, B_{kj}\ and\ C$ are $k \times k$ parameter matrices, and C is lower triangular matrix of constants. The BEKK model is covariance stationary if and only if the eigenvalues of

$$\sum_{j=1}^{q}\sum_{k=1}^{k} A_{kj} \otimes A_{kj} + \sum_{j=1}^{p}\sum_{k=1}^{k} B_{kj} \otimes B_{kj}\ have\ modulus\ less\ than\ one$$

where, $\otimes$ denotes the Kronecker product of two matrices, and are less than one in modulus. Engle and Kroner (1995) presented the Bivariate GARCH(1,1) model without diagonal restriction as

$$\begin{bmatrix}\sigma_{11,t|t-1}\\ \sigma_{12,t|t-1}\\ \sigma_{22,t|t-1}\end{bmatrix} = \begin{bmatrix}\gamma_{10}\\ \gamma_{20}\\ \gamma_{30}\end{bmatrix} + \begin{bmatrix}\gamma_{11} & \gamma_{12} & \gamma_{13}\\ \gamma_{21} & \gamma_{22} & \gamma_{23}\\ \gamma_{31} & \gamma_{32} & \gamma_{33}\end{bmatrix}\begin{bmatrix}U_{1,t-1}^2\\ U_{1,t-1}U_{2,t-1}\\ U_{2,t-1}^2\end{bmatrix} + \begin{bmatrix}g_{11} & g_{12} & g_{13}\\ g_{21} & g_{22} & g_{23}\\ g_{31} & g_{32} & g_{33}\end{bmatrix}\begin{bmatrix}\sigma_{11,t-1|t-2}\\ \sigma_{12,t-1|t-2}\\ \sigma_{22,t-1|t-2}\end{bmatrix} \quad (8)$$

The above vectorization operator includes variances and covariances of the processes. Related to multivariate GARCH models also include Kawakatsu (2006), Engle et al. (1990), Sentana (1998), Van Der (2007), Vrantos et al. (2003), Lanne and Sarkkonen (2007), Usoro and John (2019) and Usoro et al. (2020).

3 Methodology

3.1 Volatility Measure

If a process, say Y_t has an error term ϵ_t and its variance σ_t^2, then

$$\epsilon_t = \sigma_t z_t \;=>\; \sigma_t^2 = \frac{\epsilon_t^2}{z_t^2} \tag{9}$$

where, $z_t \sim iid(0,1)$, σ_t^2 is a measure of volatility, σ_t^2 follows ARCH (q) process if is a linear combination of the lagged squared error terms $\epsilon_{t-1}^2, \epsilon_{t-2}^2, \ldots \epsilon_{t-q}^2$, and GARCH(p,q) process if is a linear combination $\sigma_{t-1}^2, \sigma_{t-2}^2, \ldots, \sigma_{t-p}^2$ *and* $\epsilon_{t-1}^2, \epsilon_{t-2}^2, \ldots \epsilon_{t-q}^2$, Engle et al (1993), Gujarati and Porter (2009).

Also considered as an alternative measure of volatility is the square of the log return series of a process, say Y_t. Gujarati and Porter (2009) also expressed volatility measure as follows,

Given Y_t as time series process, $\log Y_t$ as the logarithm of Y_t,

$$\text{dlog}Y_t = \log Y_t - \log Y_{t-1} \tag{10}$$

$\text{dlog}Y_t$ is the return series. Let $\text{dlog}\overline{Y}_t$ be the mean of $\text{dlog}Y_t$.

$$X_t^2 = (\text{dlog}Y - \text{dlog}\overline{Y}_t)^2 \tag{11}$$

X_t^2 is a measure of volatility. Equations (9) and (11) are measures of volatility. In this paper, we adopt equation (11) as an alternative approach to the measure of volatility devoid of simulation of $z_t \sim iid(0,1)$. Equation 9 is the measure of volatility obtained as the square of the error component divided by the square of the standard normal random variable whose values are simulated. In this work, we adopt the volatility measure as a square of the log return series of the original data without simulation, as shown in equation (11). This is proposed to succinctly reveal the behaviour of the volatility measure through autocorrelation and partial autocorrelation functions for the choice of alternative MGARCH models.

3.2 Multivariate GARCH Models

Bollerslev et al. (1988) presented vectorization operator whose parameter matrices are limited to only principal diagonal elements. The models do not allow a greater range of interactions between the distributed lags of the response and predictor vector of conditional variances. To avert the parameter restriction, we adopt Engle and Kroner (1995) to allow free interactions of response variances.

Definition

Let $Y_{it(i=1,\ldots,m)}$ be multiple time series processes with variances $\sigma_{it(i=1,\ldots,m)}^2$, squared error terms $\epsilon_{vt(v=1,\ldots,n)}^2$ and constants $\mu_{i(i=1,\ldots,m)}$. If σ_{it-k}^2 and ϵ_{vt-s}^2 are lagged autoregressive and moving average terms such that $\sigma_{it(i=1,\ldots,m)}^2$ are functions of σ_{it-k}^2 and ϵ_{vt-s}^2 with respective matrices of parameters $\alpha_{ij.k(j=1,\ldots,n)}$ and $\beta_{iv.s(v=1,\ldots,n)}$, then $\sigma_{it(i=1,\ldots,m)}^2$ are MGARCH(p,q) models.

The models are expressed in matrix form as follows:

$$\begin{pmatrix} \sigma_{1t}^2 \\ \sigma_{2t}^2 \\ \vdots \\ \sigma_{mt}^2 \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_m \end{pmatrix} + \begin{pmatrix} \alpha_{11.1}\alpha_{12.1} \cdots \alpha_{1n.1} \\ \alpha_{21.1}\alpha_{22.1} \cdots \alpha_{2n.1} \\ \vdots \quad \vdots \quad \vdots \\ \alpha_{m1.1}\alpha_{m2.1} \ldots \alpha_{mn.1} \end{pmatrix} \begin{pmatrix} \sigma_{1t-1}^2 \\ \sigma_{2t-1}^2 \\ \vdots \\ \sigma_{nt-1}^2 \end{pmatrix}$$

$$+\begin{pmatrix} \alpha_{11.2}\alpha_{12.2}\cdots\alpha_{1n.2} \\ \alpha_{21.2}\alpha_{22.2}\cdots\alpha_{2n.2} \\ \vdots \quad \vdots \quad \vdots \\ \alpha_{m1.2}\alpha_{m2.2}\dots\alpha_{mn.2} \end{pmatrix}\begin{pmatrix} \sigma^2_{1t-2} \\ \sigma^2_{2t-2} \\ \vdots \\ \sigma^2_{nt-2} \end{pmatrix} + \cdots + \begin{pmatrix} \alpha_{11.p}\alpha_{12.p}\cdots\alpha_{1n.p} \\ \alpha_{21.p}\alpha_{22.p}\cdots\alpha_{2n.p} \\ \vdots \quad \vdots \quad \vdots \\ \alpha_{m1.p}\alpha_{m2.p}\dots\alpha_{mn.p} \end{pmatrix}\begin{pmatrix} \sigma^2_{1t-p} \\ \sigma^2_{2t-p} \\ \vdots \\ \sigma^2_{nt-p} \end{pmatrix}$$

$$+\begin{pmatrix} \beta_{11.1}\beta_{12.1}\cdots\beta_{1n.1} \\ \beta_{21.1}\beta_{22.1}\cdots\beta_{2n.1} \\ \vdots \quad \vdots \quad \vdots \\ \beta_{m1.1}\beta_{m2.1}\dots\beta_{mn.1} \end{pmatrix}\begin{pmatrix} \epsilon^2_{1t-1} \\ \epsilon^2_{2t-1} \\ \vdots \\ \epsilon^2_{nt-1} \end{pmatrix} + \begin{pmatrix} \beta_{11.2}\beta_{12.2}\cdots\beta_{1n.2} \\ \beta_{21.2}\beta_{22.2}\cdots\beta_{2n.2} \\ \vdots \quad \vdots \quad \vdots \\ \beta_{m1.2}\beta_{m2.2}\dots\beta_{mn.2} \end{pmatrix}\begin{pmatrix} \epsilon^2_{1t-2} \\ \epsilon^2_{2t-2} \\ \vdots \\ \epsilon^2_{nt-2} \end{pmatrix}$$

$$+\cdots+\begin{pmatrix} \beta_{11.q}\beta_{12.q}\cdots\beta_{1n.q} \\ \beta_{21.q}\beta_{22.q}\cdots\beta_{2n.q} \\ \vdots \quad \vdots \quad \vdots \\ \beta_{m1.q}\beta_{m2.q}\dots\beta_{mn.q} \end{pmatrix}\begin{pmatrix} \epsilon^2_{1t-q} \\ \epsilon^2_{2t-q} \\ \vdots \\ \epsilon^2_{nt-q} \end{pmatrix} \qquad (12)$$

The expansion of the above matrices is as applicable to Vector Autoregressive (VAR) models, Wikipedia (2012), Eric and Jiahui (2006), Mittnik (1989) and Chepngetich and John (2015). The pairs of matrices on the right hand side (RHS) of equation 12 are compatible in multiplication such that,

(1) $\sigma^2_{1t}$is a linear combination of $\mu_1, \sigma^2_{jt-k(j=1,\dots,n\,;k=1,\dots,p)}$ and $\epsilon^2_{vt-s(v=1,\dots,n\,;s=1,\dots,q)}$ with associated parameters $\alpha_{1j.k(j=1,\dots,n\,;k=1,\dots,p)}$ *and* $\beta_{1v.s(v=1,\dots,n\,;s=1,\dots,q)}$.

(2) $\sigma^2_{2t}$is a linear combination of $\mu_2, \sigma^2_{jt-k(j=1,\dots,n\,;k=1,\dots,p)}$ and $\epsilon^2_{vt-s(v=1,\dots,n\,;s=1,\dots,q)}$ with associated parameters $\alpha_{2j.k(j=1,\dots,n\,;k=1,\dots,p)}$ *and* $\beta_{2v.s(v=1,\dots,n\,;s=1,\dots,q)}$.

(3) $\sigma^2_{mt}$is a linear combination of $\mu_m, \sigma^2_{jt-k(j=1,\dots,n\,;k=1,\dots,p)}$ and $\epsilon^2_{vt-s(v=1,\dots,n\,;s=1,\dots,q)}$ with associated parameters $\alpha_{mj.k(j=1,\dots,n\,;k=1,\dots,p)}$ *and* $\beta_{mv.s(v=1,\dots,n\,;s=1,\dots,q)}$.

The expansion of the above matrices produces the following MGARCH models,

$$\sigma^2_{1t} = \mu_1 + \alpha_{11.1}\sigma^2_{1t-1} + \alpha_{12.1}\sigma^2_{2t-1} + \cdots + \alpha_{1n.1}\sigma^2_{nt-1} + \alpha_{11.2}\sigma^2_{1t-2} + \alpha_{12.2}\sigma^2_{2t-2} + \cdots + \alpha_{1n.2}\sigma^2_{nt-2} + \cdots + \alpha_{11.p}\sigma^2_{1t-p} + \alpha_{12.p}\sigma^2_{2t-p} + \cdots + \alpha_{1n.p}\sigma^2_{nt-p} + \beta_{11.1}\epsilon^2_{1t-1} + \beta_{12.1}\epsilon^2_{2t-1} + \cdots + \beta_{1n.1}\epsilon^2_{nt-1} + \beta_{11.2}\epsilon^2_{1t-2} + \beta_{12.2}\epsilon^2_{2t-2} + \cdots + \beta_{1n.2}\epsilon^2_{nt-2} + \cdots + \beta_{11.q}\sigma^2_{1t-q} + \beta_{12.q}\epsilon^2_{2t-q} + \cdots + \beta_{1n.q}\epsilon^2_{nt-q} \qquad (13)$$

$$\sigma^2_{2t} = \mu_2 + \alpha_{21.1}\sigma^2_{1t-1} + \alpha_{22.1}\sigma^2_{2t-1} + \cdots + \alpha_{2n.1}\sigma^2_{nt-1} + \alpha_{21.2}\sigma^2_{1t-2} + \alpha_{22.2}\sigma^2_{2t-2} + \cdots + \alpha_{2n.2}\sigma^2_{nt-2} + \cdots + \alpha_{21.p}\sigma^2_{1t-p} + \alpha_{22.p}\sigma^2_{2t-p} + \cdots + \alpha_{2n.p}\sigma^2_{nt-p} + \beta_{21.1}\epsilon^2_{1t-1} + \beta_{22.1}\epsilon^2_{2t-1} + \cdots + \beta_{2n.1}\epsilon^2_{nt-1} + \beta_{21.2}\epsilon^2_{1t-2} + \beta_{22.2}\epsilon^2_{2t-2} + \cdots + \beta_{2n.2}\epsilon^2_{nt-2} + \cdots + \beta_{21.q}\sigma^2_{1t-q} + \beta_{22.q}\epsilon^2_{2t-q} + \cdots + \beta_{2n.q}\epsilon^2_{nt-q} \qquad (14)$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$\sigma^2_{mt} = \mu_m + \alpha_{m1.1}\sigma^2_{1t-1} + \alpha_{m2.1}\sigma^2_{2t-1} + \cdots + \alpha_{mn.1}\sigma^2_{nt-1} + \alpha_{m1.2}\sigma^2_{1t-2} + \alpha_{m2.2}\sigma^2_{2t-2} + \cdots$$

$$+ \alpha_{mn.2}\sigma^2_{nt-2} + \cdots + \alpha_{m1.p}\sigma^2_{1t-p} + \alpha_{m2.p}\sigma^2_{2t-p} + \cdots + \alpha_{mn.p}\sigma^2_{nt-p} + \beta_{m1.1}\epsilon^2_{1t-1} +$$

$$\beta_{m2.1}\epsilon_{2t-1}^2 + \cdots + \beta_{mn.1}\epsilon_{nt-1}^2 + \beta_{m1.2}\epsilon_{1t-2}^2 + \beta_{m2.2}\epsilon_{2t-2}^2 + \cdots + \beta_{mn.2}\epsilon_{nt-2}^2 + \cdots +$$

$$\beta_{m1.q}\sigma_{1t-q}^2 + \beta_{m2.q}\epsilon_{2t-q}^2 + \cdots + \beta_{mn.q}\epsilon_{nt-q}^2 \quad (15)$$

The above models reduce to the form

$$\sigma_{1t}^2 = \mu_1 + \sum_{j=1}^{n}\sum_{k=1}^{p}\alpha_{1j.k}\sigma_{jt-k}^2 + \sum_{v=1}^{n}\sum_{s=1}^{q}\beta_{1v.s}\epsilon_{vt-s}^2 \quad (16)$$

$$\sigma_{2t}^2 = \mu_2 + \sum_{j=1}^{n}\sum_{k=1}^{p}\alpha_{2j.k}\sigma_{jt-k}^2 + \sum_{v=1}^{n}\sum_{s=1}^{q}\beta_{2v.s}\epsilon_{vt-s}^2 \quad (17)$$

$\vdots$ $\vdots$

$$\sigma_{mt}^2 = \mu_m + \sum_{j=1}^{n}\sum_{k=1}^{p}\alpha_{mj.k}\sigma_{jt-k}^2 + \sum_{v=1}^{n}\sum_{s=1}^{q}\beta_{mv.s}\epsilon_{vt-s}^2 \quad (18)$$

Equations 16, 17 and 18 are reduced to the form

$$\sigma_{it}^2 = \mu_i + \sum_{i,j=1}^{m,n}\sum_{k=1}^{p}\alpha_{ij.k}\sigma_{jt-k}^2 + \sum_{i,v=1}^{m,n}\sum_{s=1}^{q}\beta_{iv.s}\epsilon_{vt-s}^2 \quad (19)$$

where, $m = n$.

From equation 19

$$\sigma_{it}^2 = \mu_i + \sum_{i,j=1}^{m,n}\sum_{k=1}^{p}\alpha_{ij.k}\sigma_{jt-k}^2 \quad (20)$$

and

$$\sigma_{it}^2 = \mu_i + \sum_{i,v=1}^{m,n}\sum_{s=1}^{q}\beta_{iv.s}\epsilon_{vt-s}^2 \quad (21)$$

Equations 20 and 21 are ISO-MGARCH(p,0) models and ISO-MGARCH(0,q) models, respectively. These are Isolated Multivariate Generalised Autoregressive Conditional Heteroscedasticity models for volatility measure, whose behaviour through correlogram check is characterised by autoregressive or moving average process. These models modify Borllerslev et al. (1988).

Conditions for Model Identification:

Equation 20 is ISO-MGARCH(p,0) model of the form

$$\sigma_{it}^2 = \mu_i + \sum_{i,j=1}^{m,n}\sum_{k=1}^{p}\alpha_{ij.k}\sigma_{jt-k}^2$$

Given equation 19 above,
let $\sigma_{it}^2 = \mu + A + B$, where μ *is constant*,

$$A = \sum_{i,j=1}^{m,n} \sum_{k=1}^{p} \alpha_{ij.k} \sigma_{jt-k}^2$$

and

$$B = \sum_{i,v=1}^{m,n} \sum_{s=1}^{q} \beta_{iv.s} \epsilon_{vt-s}^2$$

$$Let\ s = 0,$$

$$=> B = \sum_{i,v=1}^{m,n} \beta_{iv.0} \epsilon_{vt-0}^2 = \sum_{i=1}^{m} \sum_{v=1}^{n} \beta_{iv.0} \epsilon_{vt-0}^2$$
$$= \sum_{i=1}^{m} [\beta_{i1}\epsilon_{1t}^2 + \beta_{i2}\epsilon_{2t}^2 + \cdots + \beta_{in}\epsilon_{nt}^2]$$
$$= \epsilon_{1t}^2[\beta_{11} + \beta_{21} + \cdots + \beta_{m1}] + \epsilon_{2t}^2[\beta_{12} + \beta_{22} + \cdots + \beta_{m2}] + \cdots$$
$$+\epsilon_{nt}^2[\beta_{1n} + \beta_{2n} + \cdots + \beta_{mn}]$$
$$= \beta_{11}\epsilon_{1t}^2 + \beta_{21}\epsilon_{1t}^2 + \cdots + \beta_{m1}\epsilon_{1t}^2 + \beta_{12}\epsilon_{2t}^2 + \beta_{22}\epsilon_{2t}^2 + \cdots + \beta_{m2}\epsilon_{2t}^2 +$$
$$\ldots + \beta_{1n}\epsilon_{nt}^2 + \beta_{2n}\epsilon_{nt}^2 + \cdots + \beta_{mn}\epsilon_{nt}^2$$

It is obvious that $s = 0 => \epsilon_{vt-0}^2 \sim iid(0, \sigma_e^2)$. Therefore, taking expectation of B becomes,

$$E(B) = E(\beta_{11}\epsilon_{1t}^2 + \beta_{21}\epsilon_{1t}^2 + \cdots + \beta_{m1}\epsilon_{1t}^2 + \beta_{12}\epsilon_{2t}^2 + \beta_{22}\epsilon_{2t}^2 + \cdots + \beta_{m2}\epsilon_{2t}^2 +$$
$$\ldots + \beta_{1n}\epsilon_{nt}^2 + \beta_{2n}\epsilon_{nt}^2 + \cdots + \beta_{mn}\epsilon_{nt}^2)$$

$= \beta_{11}E(\epsilon_{1t}^2) + \beta_{21}E(\epsilon_{1t}^2) + \cdots + \beta_{m1}E(\epsilon_{1t}^2) + \beta_{12}E(\epsilon_{2t}^2) + \beta_{22}E(\epsilon_{2t}^2) + \cdots + \beta_{m2}E(\epsilon_{2t}^2) +$

$\ldots + \beta_{1n}E(\epsilon_{nt}^2) + \beta_{2n}E(\epsilon_{nt}^2) + \cdots + \beta_{mn}E(\epsilon_{nt}^2) = 0$, since $E(\epsilon_{vt}^2) = 0$

Eliminating B (B= 0) leaves σ_{it}^2 with only autoregressive component. Hence,

$$\sigma_{it}^2 = \mu_i + \sum_{i,j=1}^{m,n} \sum_{k=1}^{p} \alpha_{ij.k} \sigma_{jt-k}^2$$

Similarly, given equation (19) above,
$k = 0 => \sigma_{jt-0}^2 = 0$. This implies the autoregressive component of the response vector of variances is uncorrelated or exhibits exponential decay, giving rise to pure moving average component of the process as applicable to ARMA process. Therefore, $\alpha_{ij.0} = 0$, implies that σ_{it}^2 are expressed only in terms of ϵ_{vt-s}^2 with $\beta_{iv.s} \neq 0$. This precludes σ_{jt-k}^2 in the expression of σ_{it}^2. Eliminating A(A= 0), leaves σ_{it}^2 with only moving average component. Hence,

$$\sigma_{it}^2 = \mu_i + \sum_{i,v=1}^{m,n} \sum_{s=1}^{q} \beta_{iv.s} \epsilon_{vt-s}^2 \text{ is } ISO - MGARCH(0, q) model.$$

3.3 Model selection criteria

Here, different model selection criteria are used to compare the performances of univariate and multivariate GARCH models. These include;
(i) Akaike Information Criterion (AIC):

$$AIC = ln\left(\frac{RSS}{n}\right) + \left(\frac{2k}{n}\right) \tag{22}$$

where, RSS = residual sum of squares, n = number of observations, k = number of parameters in the model.
(ii) Bayesian Information Criterion (BIC):

$$BIC = n \times ln\left(\frac{RSS}{n}\right) + K\{ln(n)\} \tag{23}$$

where, RSS, n and K are as defined above.

(iii) Schwarz's Information Criterion (SIC):

$$SIC = ln\left(\frac{RSS}{n}\right) + \left(\frac{k}{n}\right) \ln(n) \tag{24}$$

where, RSS, n and K are as defined above.

4. Analysis and Results

4.1 Graphical analysis

Here, is the graphical display of each of the return series. Figures 1, 2 and 3 present the trend analysis of the three return series. Each trend equation indicates stability in the series from which volatility measures are obtained.

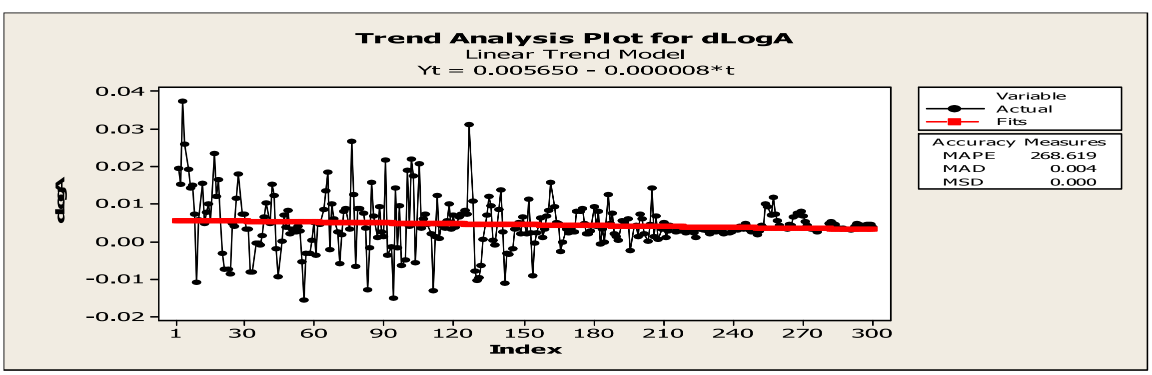

Figure 1: Return series of Average CPI

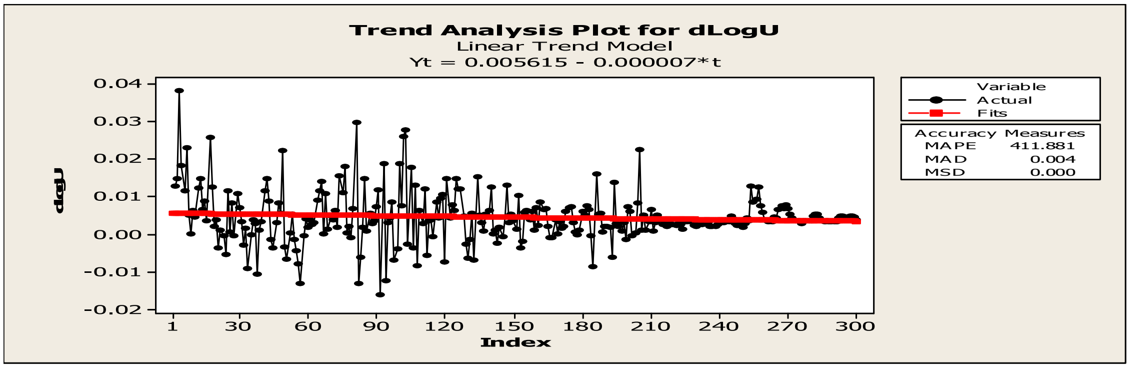

Figure 2: Return Series of Urban CPI

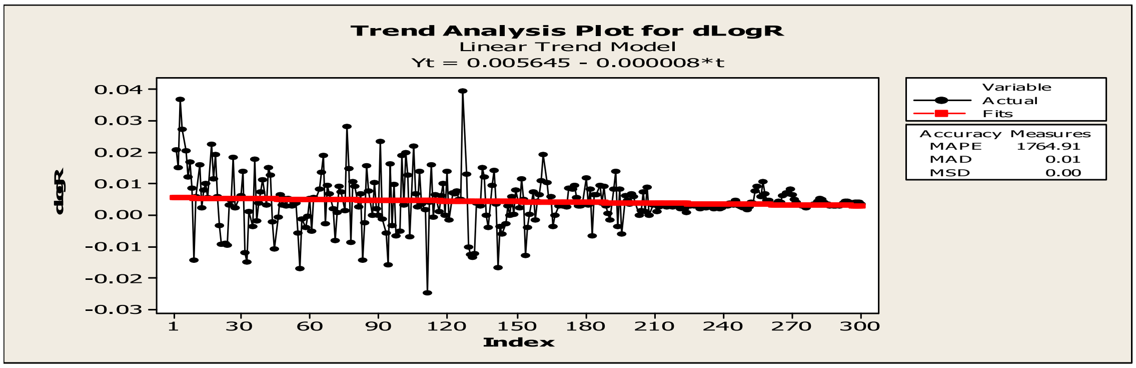

Figure 3: Return Series of Rural CPI

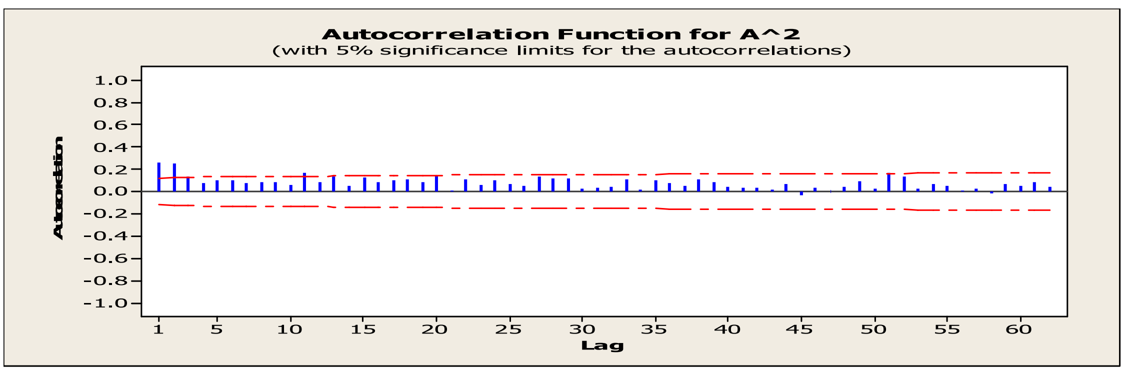

Figure 4: Autocorrelation Function of Average Consumer Price Index

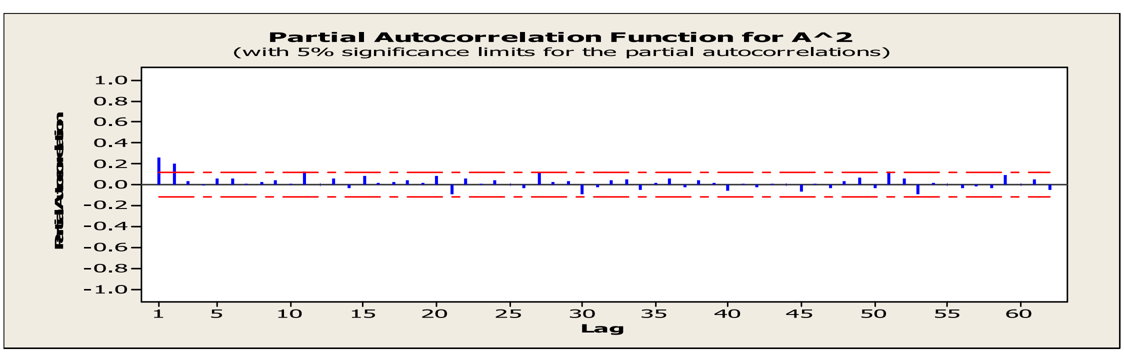

Figure 5: Partial Autocorrelation Function of Average Consumer Price Index

4.2 Univariate GARCH (p, 0) Model Estimates

The real life data for the verification of the models are three classes of Nigeria Price Consumer Index. These include the Average Consumer Price Index (ACPI), Urban Consumer Price Index (Urban CPI) and Rural Price Index (Rural CPI). The variance of the return series measures the volatility of each CPI. Firstly, we consider estimation of parameters for Univariate GARCH models for the Average CPI, Urban CPI and Rural CPI.
The estimates are obtained as follows:

Table 1: Parameter Estimates of the Univariate GARCH (2,0) Models

Predictor	Coefficients	SE. Coefficients	T	P
Average Consumer Price Index				
Constant	0.00002597	0.00000663	3.91	0.000
σ^2_{1t-1}	0.20799	0.05714	3.64	0.000
σ^2_{1t-2}	0.20001	0.05688	3.52	0.001
Urban Consumer Price Index				
Constant	0.00003008	0.00000673	4.47	0.000
σ^2_{2t-1}	0.24917	0.05822	4.28	0.000
σ^2_{2t-2}	0.05253	0.05823	0.90	0.368
Rural Consumer Price Index				
Constant	0.00004036	0.00000906	4.46	0.000
σ^2_{3t-1}	0.14877	0.057472	2.59	0.010
σ^2_{3t-2}	0.17107	0.05726	2.99	0.003

The Analysis in Table1 provides parameter estimates for the univariate GARCH models for the conditional variances of the Average, Urban and Rural CPI represented by σ^2_{1t}, σ^2_{2t} *and* σ^2_{3t} respectively. The order of the GARCH models is suggested on the basis of the ACF and PACF in Figures 4 and 5. The estimated parameters are significant for each of the Isolated GARCH Models for the pure autoregressive process.

4.3 Isolated Multivariate GARCH (p,0) Model Estimates

Table 2 gives estimates of the multivariate GARCH models with response vector conditional variances of Average, Urban and Rural Consumer Price Indices. The choice of the order of the models is as applicable to the univariate GARCH models whose parameter estimates are in Table1.

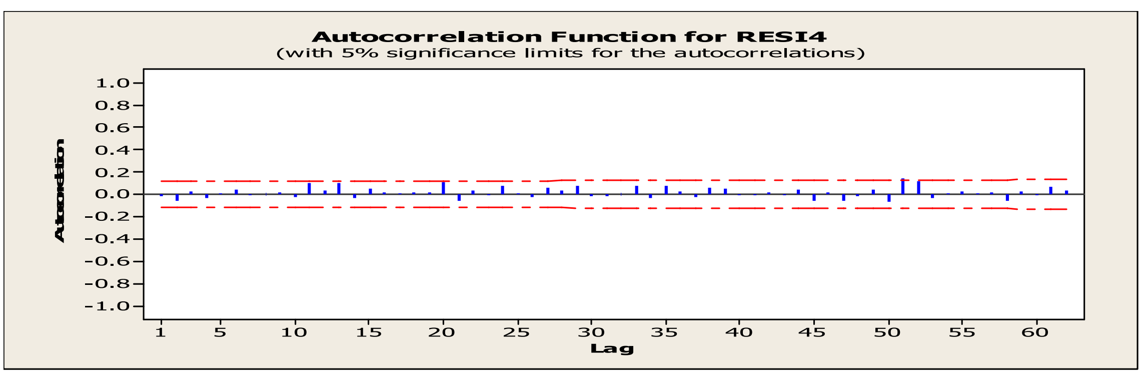

Figure 6: Autocorrelation function the residual of σ^2_{1t}MGARCH model

Table 2: **Parameter Estimates of the Multivariate GARCH(2,0) Models**

Predictor	Coefficients	SE. Coefficients	T	P
Average Consumer Price Index				
Constant	0.00002102	0.00000689	3.05	0.003
σ^2_{1t-1}	0.0765	0.1811	0.42	0.673
σ^2_{1t-2}	0.0160	0.1797	0.09	0.929
σ^2_{2t-1}	0.15513	0.07689	2.02	0.045
σ^2_{2t-2}	0.17229	0.07743	2.23	0.027
σ^2_{3t-1}	0.0163	0.1202	0.14	0.892
σ^2_{3t-2}	0.0641	0.1198	0.54	0.593
Urban Consumer Price Index				
Constant	0.00002642	0.00000713	3.70	0.000
σ^2_{1t-1}	-0.0259	0.1874	-0.14	0.890
σ^2_{1t-2}	0.0778	0.1860	0.42	0.676
σ^2_{2t-1}	0.22871	0.07956	2.87	0.004
σ^2_{2t-2}	-0.01529	0.08012	-0.19	0.849
σ^2_{3t-1}	0.0566	0.1244	0.45	0.650
σ^2_{3t-2}	0.0285	0.1240	0.23	0.818
Rural Consumer Price Index				
Constant	0.00003225	0.00000923	3.49	0.001
σ^2_{1t-1}	-0.0514	0.2425	-0.21	0.832
σ^2_{1t-2}	0.0543	0.2406	0.23	0.822
σ^2_{2t-1}	0.2040	0.1029	1.98	0.048
σ^2_{2t-2}	0.2147	0.1037	2.07	0.039
σ^2_{3t-1}	0.0930	0.1610	0.53	0.564
σ^2_{3t-2}	0.0563	0.1604	0.35	0.726

The autocorrelation function in Figure 6 exhibits pure white noise process of the residual term. This precludes moving average part of the model since the volatility measures are accounted for by autoregressive process only both in the univariate and multivariate GARCH models.

Table 3: Information Criteria

S/N	Model Specification	Response Variance	AIC	BIC	SIC
1	GARCH (2,0)	σ^2_{1t}	-18.4042	-5477.05	-18.3794
2	GARCH (2,0)	σ^2_{2t}	-18.3615	-5464.35	-18.3367
3	GARCH (2,0)	σ^2_{3t}	-17.8064	-5298.92	-17.7816
4	MGARCH (2,0)	σ^2_{1t}	-18.4156	-5465.67	-18.3412
5	MGARCH (2,0)	σ^2_{2t}	-18.3474	-5445.35	-18.2730
6	MGARCH (2,0)	σ^2_{3t}	-17.8321	-5291.79	-17.7577

Table 4 contains values of model information selection criteria of equation 22, 23 and 24. The univariate and multivariate GARCH models compete favourably in each response variance. The two sets of models have the same comparative advantage in capturing volatility measure of each CPI.

5 Summary and Conclusion

The major focus of this research was to identify some classes of Multivariate GARCH models, considering the behaviour of the return series and the volatility measures of the Nigeria Consumer Price Indices. The time plot of each of the return series exhibited volatility clustering as indicated in Figures 1, 2 and 3. This was followed with the autocorrelation and partial autocorrelation functions of the volatility series displayed in Figures 4 and 5. The ACF and PACF featured the behaviour of the volatility series and is accounted for by a pure autoregressive process. This explained the reason for Isolated Multivariate GARCH(p,0) Model. From the estimated models, the order p = 2 is obtained from the PACF of the volatility measure. Ordinarily, GARCH(p,q) model is a generalized model and of course an extension of ARCH(q) model in similar form with ARMA(p,q) model (where p and q represent the order of autoregressive and moving average processes), from which each of the processes is isolated on certain conditions. The idea of identifying MGARCH(p.0) and MGARCH(0.q), where p and q represent the order of each isolated process is under a certain condition that there exists some volatility measures accounted for by either autoregressive, moving average or both processes as in the case of Autoregressive Moving Average (ARMA) Process and Bilinear Autoregressive Moving Average (BARMA) Process.

From the Multivariate Generalized Conditional Autoregressive Heteroscedasticity MGARCH(p,q) model, Isolated Multivariate Generalized Conditional Autoregressive Heteroscedasticity, ISO-MGARCH(p,0) and Isolated Multivariate Generalized Conditional Autoregressive Heteroscedasticity, ISO-MGARCH(0,q) have been identified for volatility measures accounted for by a single stationary autoregressive or moving average process. Under certain conditions the isolated models 20 and 21 have been developed. Furtherance to verification, parameters of the identified models are estimated with volatility measures of Nigeria CPI data from January 1995 to December 2019. Table 2 displays parameter estimates of the Isolated Multivariate Generalized Conditional Autoregressive Heteroscedasticity Models. Figure 6 is the autocorrelation function of the residual of the fitted models, indicating a pure white noise process. Hence, the models are appropriate for volatility measures depending on the dynamic nature of each economic or financial time series. Notwithstanding the fact that a number of parameters of the ISO-MGARCH(2,0) models are not significant, the multiple parameter models have given way for possible interactions, hence, the reason for significant contributions of urban price index lagged variance to average and rural price indices, which is the advantage of allowing large range of interactions between each response and other predictor variances. Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC) and Schwarz's Information Criterion (SIC) results in Table 3 have revealed same comparative advantage of the isolated univariate and multivariate GARCH models from the empirical data. The validity of the ISO-MGARCH model is evident in the autocorrelation function in Figure 6. The existing gap between this paper and previous researches is the identification of isolated univariate and multivariate GARCH models from the generalized form whose response vector of variances (volatility measures) could be expressed as linear combinations of its distributed lagged variance from the original series precluding its dependence on the squared errors and simulated values of the standard normal random variable (z_t). As an alternative to the existing approach, ISO-MGARCH models require the behavioural structures of the ACF and PACF of the raw response variance like in the case of ARMA/ARIMA process.

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