Class of Generalized Power Function Distributions: Properties and Applications

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There are different frameworks available in theory used in developing convoluted distributions. One of the unique frameworks is the T-R{Y} framework. The T-R{Y} framework is used to generate distributions that have more than one mode, and the developed distributions are weighted hazard function of the base line distributions. The two parameter Power function distribution is a useful distribution that has the properties of Pareto, Weibull, Uniform, Kumaraswamy, and Beta distributions. The combination of gamma and power functions distributions with both having shape parameters and the upper bound parameter of power function distribution will produce a distribution, which is more flexible than gamma and power function distributions. The new distribution will present new opportunities for assessing reliability and survival data in different field of study. This research however, developed and studied the properties of T-Power{Y} family of distributions using the T-R{Y} framework. A special case of this family was developed as the Gamma-Power {log-logistic} distribution (GPLD), and was characterized using different functions and different properties of the distribution were derived. We used Maximum Likelihood Estimation (MLE) method to estimate the distribution parameters, except for that of the upper bound. Simulation study was carried out to test the consistency of the parameters estimates and was applied to two real life data. The results of the new distribution were compared with existing distributions, and the comparison shows that the proposed GPLD performed best when compared with some existing distributions.

Keywords: convoluted distribution; gamma-power distribution; power function distribution; quantile function; mean deviation;*T*-R{Y} framework

1 Introduction

Developing distributions by combining two or more distributions to form new ones is trending in the field of probability and statistics. Combining the flexibility of gamma, power function, and log-logistic distributions, will provide distribution with greater flexibility. The power function distribution is a flexible lifetime distribution, which may be obtained through simple transformation of the Pareto, beta, kumaraswamy and uniform distributions (Dallas 1976).The usefulness of the power function distribution can be found in the work of Meniconi and Barry (1996), Ali and Woo (2005), Johnson *et al.* (1994); Balakrishnan and Nevzorov (2003); Kleiber and Kotz (2003); Forbes et al.(2011); Zaka and Akhter (2013); Zaka et al.(2013); Asghar et al. (2013); Ali and Woo (2005); Sinha et al. (2008); Ali *et al*. (2000). Convoluted distribution as the combination of two or more distributions to form a

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new distribution or any transformation done to an existing distribution to form a new distribution (Akarawak *et al*., 2013). The new distribution is a hybrid, which is expected to perform better than the individual distributions.

Aljarrah el al., (2014) proposed the *T*-Weibull{Y} family and *T*-Uniform{Y} family, and generated Normal-Weibull {Cauchy}, Normal-Weibull {logistic}, Weibull-Uniform {log-logistic} distributions. Alzaatreh el al. (2014) proposed the *T*-Normal{Y} family, and generated Weibull-Normal {exponential}, Exponential-Normal {Log-logistic}, Logistic-Normal{Logistic} and Logistic-Normal{Extreme value} distributions. Famoye et al. (2018) also generated Weibull-Normal {log-logistic} distribution, which is a member from the *T*-Normal{Y} family. Alzaatreh el al. (2016) proposed the *T*-Cauchy{Y} family, and generated Weibull-Cauchy {Exponential}, Lomax-Cauchy {Log-logistics}, Gamma-Cauchy {Extreme value} distributions. Zubair el al. (2018) proposed the *T*-Exponential{Y} family, and generated Weibull-Exponential {Log-logistic}, Gamma-Exponential {Log-logistic}, Normal-Exponential {Log-logistic}. Recently, Ogunsanya et al. (2021) developed the Weibull-inverse Rayleigh distribution using the T-R{Y} framework.

The *T*-R{Y} family can be increasing, decreasing, upside-down bathtub and bathtub shaped density functions. Due to great flexibility of its hazard rate functions, it thus provides a good alternative to many existing lifetime distributions. More so, combining three flexible distributions will provide greater flexibility. It is observed that many real life data are bounded above. For instance, scores in percentage are bounded above by 100, probability values are bounded above by 1, human age has a maximum upper limit, pH scale has a maximum of 7, and so on. The idea of 0 to infinity does not exist in real data. It is just a mere mathematical assumption. Thus, developing distributions with upper bound should be of interest. The *T*-R{Y} family of distributions have not exploit the possibility of distributions with upper bound.

The *T*-R{Y} framework metamorphosed from Beta-X by (Eugene et al. 2002) to T-X by (Alzaatreh et al. 2013), and further extended to T-X(W) by (Aljarrah et al, 2014). Alzaatreh el al. (2014) unified the T-X family to a more generalized framework, *T-R*{*Y*} and many distributions have been derived from this family. Take *T*, *R* and *Y* to be random variables with known cumulative distribution functions (cdfs), $F_T(x)$, $F_R(x)$ and $F_Y(x)$ respectively. Also, let $f_T(x)$, $f_R(x)$ and $f_Y(x)$ be their corresponding probability density functions (pdfs) with known quantile functions $Q_T(x)$, $Q_R(x)$ and $Q_Y(x)$ respectively. Then the cumulative distribution function (cdf) of the T-R{Y} family of distributions is given by

$$F_X(x) = \int_a^{Q_Y[F_R(x)]} f_T(t)dt = P\{T \le Q_Y[F_R(x)]\} = F_T\{Q_Y[F_R(x)]\}, \tag{1}$$

where $f_T(t)$ is the pdf of a random variable *T*, $Q_Y[.]$ is the quantile function of a random variable, *Y*, and $F_R(x)$ is the cumulative distribution function (cdf) of a random variable, *R*. $Q_Y[F_R(x)]$ is differentiable and monotonically non-decreasing. It is necessary that $Q_Y[F_R(x)]$ and $f_T(t)$ have the same support. The pdf of *T-R*{*Y*} given in (1) as defined by Alzaatreh et al. (2014) is given by

$$f_X(x) = f_R(x)\frac{f_T\{Q_Y[F_R(x)]\}}{f_Y\{Q_Y[F_R(x)]\}}. \quad (2)$$

In this article, we a class of generalized power function distributions, the *T*-Power{*Y*} family, and studied some its properties and applications. A special case of the T-power {*Y*} family is the Gamma-Power function distribution (GPLD). It was studied in detail and applied to two real datasets. The new distribution will combine the usefulness and flexibility of gamma, power function and log-logistic distributions, and will be very useful in modelling survival data, especially data that are highly skewed, data with gaps, highly peaked data, and possibly bimodal data. The proposed GPLD will be helpful in modelling bimodal data, data that are highly skewed, and leptokurtic.

2 Developing the *T*-Power Function{*Y*} Distribution

Proposition 1: The pdf in (2) is a proper pdf, that is, the integral of the pdf in (2)

is equal to 1.

Proof:

$$\int_a^\infty f_X(x)dx = \int_a^\infty f_R(x)\frac{f_T\{Q_Y[F_R(x)]\}}{f_Y\{Q_Y[F_R(x)]\}}dx$$

From equation (2) we have

$$\frac{f_X(x)}{f_R(x)} = \frac{f_T\{Q_Y[F_R(x)]\}}{f_Y\{Q_Y[F_R(x)]\}}$$

So that

$$\int_a^\infty f_X(x)dx = \int_a^\infty f_R(x)\frac{f_X(x)}{f_R(x)}dx$$

$$\int_a^\infty f_X(x)dx = 1$$

The survival, hazard and cumulative hazard functions corresponding to the cdf in equation (1) are given in equations (3), (4) and (5) respectively.

$$S_X(x) = 1 - F_T\{Q_Y[F_R(x)]\} \quad (3)$$

$$h_X(x) = \frac{f_R(x)f_T\{Q_Y[F_R(x)]\}}{f_Y\{Q_Y[F_R(x)]\}(1-F_T\{Q_Y[F_R(x)]\})} \quad (4)$$

$$H_X(x) = -ln[1 - F_T\{Q_Y[F_R(x)]\}]. \quad (5)$$

The quantile function is given by

$$Q_X(p) = Q_R\{F_Y[Q_T(p)]\},\ p \in (0, 1). \quad (6)$$

Table 1: **The pdf and cdf for some *Y***

SN	Family of Distributions	Pdf	Cdf
1	*T*-R{*exponential*}	$f_X(x) = h_R(x) f_T[H_R(x)]$	$F_X(x) = F_T[H_R(x)]$
2	*T*-R{*log-logistic*}	$f_X(x) = \frac{f_R(x)}{[S_R(x)]^2} f_T\left[\frac{F_R(x)}{S_R(x)}\right]$	$F_X(x) = F_T\left[\frac{F_R(x)}{S_R(x)}\right]$
3	*T*-R{*frechet*}	$f_X(x) = \frac{f_R(x)}{F_R(x)\{ln[F_R(x)]\}^2} f_T\left\{-\frac{1}{ln[F_R(x)]}\right\}$	$F_X(x) = F_T\left\{-\frac{1}{ln[F_R(x)]}\right\}$
4	*T*-R{*logistic*}	$f_X(x) = \frac{h_R(x)}{F_R(x)} f_T\left\{ln\left[\frac{F_R(x)}{S_R(x)}\right]\right\}$	$F_X(x) = F_T\left\{ln\left[\frac{F_R(x)}{S_R(x)}\right]\right\}$
5	*T*-R{*extreme value*}	$f_X(x) = \frac{h_R(x) f_T(-ln\{-ln[F_R(x)]\})}{-F_R(x) ln[F_R(x)]}$	$F_X(x) = F_T(-ln\{-ln[F_R(x)]\})$
6	*T*-R{*uniform*}	$f_X(x) = f_R(x) f_T[F_R(x)]$	$F_X(x) = F_T[F_R(x)]$

The pdfs and cdfs for some choices of *Y* random variables are displayed on Table 1.Note that Table 1 only displayed the standard form of the distribution. This implies that the parameters of random variable *Y* will be silent in the proposed distributions. The parameters that will be involved in the proposed distributions are parameters of *T* and *R* only.

2.1 *T*-Power function{*Y*} classes of distributions

Let *R* be a random variable that follows a power function distribution with cdf, $F_R(x) = x^k/\lambda^k$ and pdf, $f_R(x) = (k/\lambda^k)x^{k-1}$. The cdf of *T*-Power function{*Y*} or simply *T*-P{*Y*} family of distribution is derived by inserting $F_R(x) = x^k/\lambda^k$ into equation (1) and it is given as

$$F_X(x) = \int_a^{Q_Y\left(\frac{x^k}{\lambda^k}\right)} f_T(t)\, dt = F_T\left[Q_Y\left(\frac{x^k}{\lambda^k}\right)\right]. \quad (7)$$

The pdf corresponding to equation (7) is derived by inserting $F_R(x) = x^k/\lambda^k$ and $f_R(x) = (k/\lambda^k)$ x^{k-1}into equation (2) and it is given as

$$f_X(x) = \frac{k}{\lambda^k} x^{k-1} \frac{f_T\left[Q_Y\left(\frac{x^k}{\lambda^k}\right)\right]}{f_Y\left[Q_Y\left(\frac{x^k}{\lambda^k}\right)\right]}. \quad (8)$$

The hazard rate function for *T*-P{*Y*} family is given as

$$h_X(x) = \frac{kx^{k-1}}{\lambda^k\left\{1-F_T\left[Q_Y\left(\frac{x^k}{\lambda^k}\right)\right]\right\}} \frac{f_T\left[Q_Y\left(\frac{x^k}{\lambda^k}\right)\right]}{f_Y\left[Q_Y\left(\frac{x^k}{\lambda^k}\right)\right]} = \frac{kx^{k-1} h_T\left[Q_Y\left(\frac{x^k}{\lambda^k}\right)\right]}{\lambda^k f_Y\left[Q_Y\left(\frac{x^k}{\lambda^k}\right)\right]}. \quad (9)$$

Remark 1. If *X* is *T*-P{*Y*} distributed, then it follows that

(i) $X = \lambda[F_Y(T)]^{1/k}$, in distribution,

(ii) $Q_X(p) = \lambda\{F_Y[Q_T(p)]\}^{1/k}$,

(iii) If $T = Y$ in distribution, then $X = Power\ function(k)$in distribution, and

(iv) If $Y = Power\ function(k)$in distribution, then $X = T$ in distribution.

The *T*-P{*Y*} family in equation (8) can be used to generate many different classes of power function distribution families. Some generalized power function families using log-logistic quantile function displayed on Table 1 are generated.

The *T*-P{*exponential*} Family

Let $T\in[0, \infty)$ be any random variable. By substituting $h_R(x)$ and $H_R(x)$ into item 1 of Table 1, *T*-R{*exponential*}, the cdf and pdf of *T*-P{*exponential*} family are respectively given by

$$F_X(x) = F_T\left[-ln\left(\lambda^k - x^k\right)\right] \tag{10}$$

and

$$f_X(x) = \frac{kx^{k-1}}{\lambda^k\left(\lambda^k - x^k\right)} f_T\left[-ln\left(\lambda^k - x^k\right)\right] \tag{11}$$

The *T*-P{*log-logistic*} Family

Let $T\in[0, \infty)$ be any random variable. By substituting $F_R(x)$ and $S_R(x)$ into item 2 of Table 1, *T*-R{*log-logistic*}, the cdf and pdf of *T*-P{*log-logistic*} family are respectively given by

$$F_X(x) = F_T\left(\frac{x^k}{\lambda^k - x^k}\right) \tag{12}$$

and

$$f_X(x) = \frac{k\lambda^k x^{k-1}}{\left(\lambda^k - x^k\right)^2} f_T\left(\frac{x^k}{\lambda^k - x^k}\right) \tag{13}$$

The *T*-P{*frechet*} Family

Let $T\in[0, \infty)$ be any random variable. By substituting $F_R(x)$ and $f_R(x)$ into item 3 of Table 1, *T*-R{*frechet*}, the cdf and pdf of *T*-P{*frechet*} family are respectively given by

$$F_X(x) = sF_T\left[-\frac{1}{ln\left(\frac{x^k}{\lambda^k}\right)}\right] \tag{14}$$

and

$$f_X(x) = \left\{\frac{kx^{k-1}}{x^k\left[ln\left(\frac{x^k}{\lambda^k}\right)\right]^2}\right\} f_T\left[-\frac{1}{ln\left(\frac{x^k}{\lambda^k}\right)}\right] \tag{15}$$

The *T*-P{*exponential*}, *T*-P{*log-logistic*} and *T*-P{*frechet*} will take care of any *T* supported on the interval $[0, \infty)$, like the Rayleigh, gamma, Weibull, Lomax and Dagum. This is because log-logistic and frechet distributions are supported on the interval $[0, \infty)$.

The *T*-P{*logistic*} Family

Let $T\in(-\infty, \infty)$ be any random variable. By substituting $F_R(x)$ and $h_R(x)$ into item 4 of Table 1, *T*-R{*logistic*}, the cdf and pdf of *T*-P{*logistic*} family are respectively given by

$$F_X(x) = F_T\left[ln\left(\frac{x^k}{\lambda^k - x^k}\right)\right] \tag{16}$$

and

$$f_X(x) = \left[\frac{kx^{k-1}}{x^k\left(\lambda^k - x^k\right)}\right] f_T\left[ln\left(\frac{x^k}{\lambda^k - x^k}\right)\right] \tag{17}$$

The *T*-P{*extreme value*} Family

Let $T \in (-\infty, \infty)$ be any random variable. By substituting $H_R(x)$ and $h_R(x)$ into item 5 of Table 1, *T*-R{*extreme value*}, the cdf and pdf of *T*-P{*extreme value*} family are respectively given by

$$F_X(x) = F_T[-ln(-klnx)] \tag{18}$$

and

$$f_X(x) = \left[\frac{kx^{k-1}}{-x^k(\lambda^k - x^k)\ln\left(\frac{x^k}{\lambda^k}\right)}\right] f_T\left\{-\ln\left[-\ln\left(\frac{x^k}{\lambda^k}\right)\right]\right\} \tag{19}$$

The *T*-P{*logistic*} and *T*-P{*extreme value*}will take care of any *T* supported on the open interval $(-\infty, \infty)$, like the normal, Cauchy, Laplace, Gumbel etc. This is because logistic distribution is also supported on the interval $(-\infty, \infty)$.

The *T*-P{*uniform*} Family

Let $T \in [0, \infty)$ be any random variable. By substituting $F_R(x)$ and $f_R(x)$ into item 6 of Table 1, *T*-R{*uniform*}, the cdf and pdf of *T*-P{*uniform*} family are respectively given by

$$F_X(x) = F_T\left(\frac{x^k}{\lambda^k}\right) \tag{20}$$

and

$$f_X(x) = \frac{k}{\lambda^k} x^{k-1} f_T\left(\frac{x^k}{\lambda^k}\right) \tag{21}$$

The *T*-P{*uniform*} will take care of any *T* supported on the closed interval [0, 1], like the beta, Kumaraswamy etc. This is because standard uniform distribution is supported on the closed interval [0, 1].

2.2 General properties of *T*-P{*Y*} family

We investigated some properties of *T*-P{*Y*}in this sub-section. We give the following Lemma to establish the relationships between *X* and *T*in order to simulate variates of *X* from the variates of *T*.

Lemma 1 (Useful Transformation). *If T is a random variate from pdf $f_T(x)$, then random variate*

(i) $X = \lambda[1 - \exp(-T)]^{1/k}$follows the *T*-P{*exponential*}family of distribution in equation (11), provided *T* is supported on the interval 0 to ∞, i.e, $T \in [0,\infty)$. The exponential parameter, rate = 1.

(ii) $X = \lambda\left(\frac{T}{1+T}\right)^{1/k}$ follows the *T*-P{*log-logistic*}family of distribution in equation (13), provided *T* is supported on the interval 0 to ∞, i.e, $T \in [0,\infty)$. The log-logistic parameters, scale = shape = 1.

(iii) $X = \exp\{\lambda[-(T^{-1})]^{1/k}\}$ follows the *T*-P{*frechet*}family of distribution in equation (15), provided *T* is supported on the interval 0 to ∞, i.e, $T \in [0,\infty)$. The frechet parameters, scale = shape = 1 and location = 0.

(iv) $X = \lambda\left(\frac{e^{-T}}{1+e^{-T}}\right)^{1/k}$ follows the T-P{$logistic$}family of distribution in equation (17), provided T is supported on the open interval -∞ to ∞, i.e., T∈(-∞,∞).The logistic parameter, scale = 1.

(v) $X = \lambda\{1 - exp[-\exp(T)]\}^{1/k}$follows the T-P{$extreme$ $value$}family of distribution in equation (19), provided T is supported on the open interval -∞ to ∞, i.e, T∈(-∞,∞). The extreme value parameters, scale = 1 and location = 0.

(vi) $X = \lambda T^{1/k}$follows the T-P{$uniform$}family of distribution in equation (21), provided T is supported on the closed interval 0 to 1, i.e., T∈[0,1].

Proof:

It result is from Remark 1(i).

Lemma 1 will be very useful in deriving some members of the generalized power function distributions. All existing univariate continuous probability distributions must satisfy at least one of the 5 situations in Lemma 1. We can generate random variate X if T is already generated.

Lemma 2 (Quantile functions). *It follows from Lemma 1 that the quantile functions of (i) T-P{log-logistic}, (ii) T-P{frechet}, (iii) T-P{logistic}, (iv) T-P{exreme value} and (v) T-P{uniform}families are given respectively by:*

(i) $Q_X(p) = \lambda\{1 - exp[-Q_T(p)]\}^{1/k}$,

(ii) $Q_X(p) = \lambda\left(\frac{Q_T(p)}{1+Q_T(p)}\right)^{1/k}$,

(iii) $Q_X(p) = \exp\left(\lambda\{-[Q_T(p)]^{-1}\}^{1/k}\right)$,

(iv) $Q_X(p) = \lambda\left(\frac{e^{-Q_T(p)}}{1+e^{-Q_T(p)}}\right)^{1/k}$,

(v) $Q_X(p) = \lambda\{1 - exp[-\exp(Q_T(p))]\}^{1/k}$,

(vi) $Q_X(p) = \lambda[Q_T(p)]^{1/k}$.

Proof:

It is easy to see the result from Remark 1(ii).

Lemma 2 will be very useful in deriving the quantile functions of the generalized power function distributions. With this, we can easily derive the median and other measures of partition.

Mode:

Theorem 1. *The modes of the T-P(log-logistic) class are the solutions of the equation*

$$x = \lambda\left[\frac{(1-k)(\lambda^k - x^k)f_T\left(\frac{x^k}{\lambda^k - x^k}\right)}{\lambda^k k^3 f_T'\left(\frac{x^k}{\lambda^k - x^k}\right) + (k+1)(\lambda^k - x^k)f_T\left(\frac{x^k}{\lambda^k - x^k}\right)}\right]^{\frac{1}{k}}$$

The mode of T-P(log-logistic) is not unique and it exist for k> 1. This suggests that it has more than one mode.

3 Gamma-Power{log-logistic} distribution (GPLD)

Most generalized power function distributions as well as Gamma distribution have been proposed in literature to provide better fitting of certain data sets than the traditional two parameters power function distribution or two parameters Gamma distribution.

3.1 Cumulative distribution and probability density functions of GPLD

In this section, we present the GPLD distribution, series expansion of its pdf, some of its sub-models, its hazard and reverse hazard functions. Some of its graphs are also presented in this section. The cumulative distribution function (cdf) of GPLD is derived by inserting the cdf of gamma distribution into equation (12) and the probability density function (pdf) is derived by inserting the pdf of gamma distribution into equation (13). The cdf of GPLD with four parameters α, β, k, λ is given by

$$F_X(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^{\frac{x^k}{\lambda^k - x^k}} t^{\alpha-1} e^{-\beta t}\, dt = \frac{1}{\Gamma(\alpha)} \gamma\left[\beta\left(\frac{x^k}{\lambda^k - x^k}\right), \alpha\right]; \alpha, \beta, k, \lambda > 0, 0 \le x \le 1 \quad (22)$$

where $\alpha$and k are shape parameters, β and λ are the scale parameter and $\gamma(x, \alpha) = \int_0^x t^{\alpha-1} e^{-t}\, dt$is the lower incomplete gamma function.
The corresponding pdf is derived by differentiating the cdf in (22) and it is given by

$$f_X(x) = \frac{k\lambda^k \beta^\alpha x^{\alpha k-1}}{\Gamma(\alpha)(\lambda^k - x^k)^{\alpha+1}} exp\left[-\beta\left(\frac{x^k}{\lambda^k - x^k}\right)\right]; \alpha, \beta, k, \lambda > 0, 0 \le x \le \lambda \quad (23)$$

The pdf plot of GPLD for some parameter values when λ = 1 are given in Figure 1. The plots depict that that the GPLD pdf can be decreasing, increasing, right or left skewed. The GD distribution has a positive and negative asymmetry. A random variable, X, that follows GPLD distribution can be written as X~GPLD(α, β, k, λ).

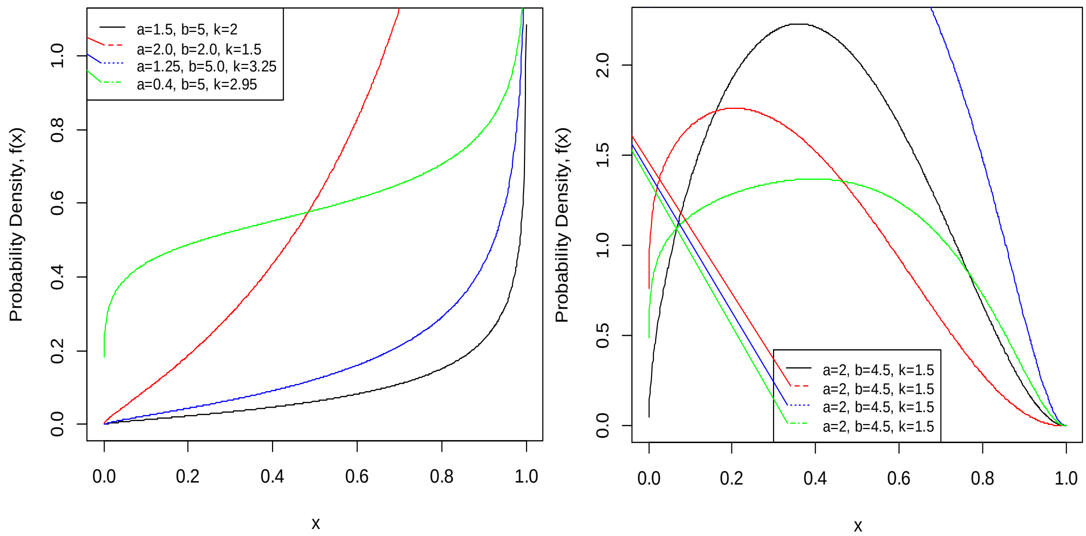

Figure 1: Graph of GPLD pdf for Some Parameters Values, λ = 1

Theorem 2: Let X be a random variable that follows a GPLD(α, β, k, λ), the series expansion of the density function of X, is a weighted power function density with parameter αk and λ. and it is given by

$$f_X(x) = \frac{\alpha!\,k\beta^{\alpha}}{\Gamma(\alpha)\lambda^{\alpha k}}x^{\alpha k-1};\alpha,\beta,k,\lambda > 0, 0 \le x \le \lambda$$

where $\frac{\alpha k}{\lambda^{\alpha k}}x^{\alpha k-1}$ is the pdf of power function distribution with shape parameter αk and scale parameterλ.

Proof:

Recall the pdf of GPLD given in equation (23) as

$$f_X(x) = \frac{k\lambda^k\beta^{\alpha}x^{\alpha k-1}}{\Gamma(\alpha)\left(\lambda^k - x^k\right)^{\alpha+1}} exp\left[-\beta\left(\frac{x^k}{\lambda^k - x^k}\right)\right]$$

The series expansion of the exponential is given by

$$\exp\left(\frac{x^k}{\lambda^k - x^k}\right) = \sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\left(\frac{x^k}{\lambda^k - x^k}\right)^i$$

$$f_X(x) = \frac{k\lambda^k\beta^{\alpha}x^{\alpha k-1}}{\Gamma(\alpha)\left(\lambda^k - x^k\right)^{\alpha+1}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\left(\frac{x^k}{\lambda^k - x^k}\right)^i$$

$$f_X(x) = \frac{k\lambda^k\beta^{\alpha}}{\Gamma(\alpha)}\sum_{i=0}^{\infty}\frac{(-1)^i x^{\alpha k+ki-1}}{i!\,\lambda^{\alpha k+ki-k}}\left(1 - \frac{x^k}{x^k}\right)^{-(\alpha+i+1)}$$

Also, we have another series expansion given by

$$\left(1 - \frac{x^k}{x^k}\right)^{-(\alpha+i+1)} = \sum_{j=0}^{\infty}\frac{(\alpha+i+j)!}{j!}\left(\frac{x^k}{x^k}\right)^j$$

$$f_X(x) = \frac{k\lambda^k\beta^{\alpha}}{\Gamma(\alpha)}\sum_{i=0}^{\infty}\frac{(-1)^i x^{\alpha k+ki-1}}{i!\,\lambda^{\alpha k+ki-k}}\sum_{j=0}^{\infty}\frac{(\alpha+i+j)!}{j!}\left(\frac{x^k}{\lambda^k}\right)^j$$

The pdf of GPLD can be written as

$$f_X(x) = \frac{k\beta^{\alpha}}{\Gamma(\alpha)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^i(\alpha+i+j)!\,x^{k(\alpha+i+j)-1}}{i!\,j!\,\lambda^{k(\alpha+i+j)}};\alpha,\beta,k,\lambda > 0, 0 \le x \le \lambda \tag{24}$$

If $i = j = 0$, GPLD pdf reduces to a density function which is a weighted power function distribution with shape parameter, ak, and scale parameter, λ. And this completes the proof.

$$f_X(x) = \frac{\alpha!\,k\beta^{\alpha}}{\Gamma(\alpha)\lambda^{\alpha k}}x^{\alpha k-1};\alpha,\beta,k,\lambda > 0, 0 \le x \le \lambda$$

$$f_Z(z) = \frac{(\alpha-1)!\,\beta^{\alpha}}{\Gamma(\alpha)}f(z);\alpha,\beta,k,\lambda > 0, 0 \le z \le \lambda \tag{25}$$

where $\frac{(\alpha-1)!\beta^{\alpha}}{\Gamma(\alpha)}$ is the weight and f(z) is the pdf of power function distribution with parameter αk, λ. Equation (25) completes the proof. Also, from Equation (25), if $\beta = \alpha = 1$, then we have power function distribution with parameters k and λ given by

$$f_Z(z) = \frac{k}{\lambda^k} z^{k-1}; k, \lambda > 0, 0 \le z \le \lambda$$

Lemma 3: Let X be a continuous random variable that follows a GPLD(α, β, k, λ), a random variable W follows a weighted gamma distribution with parameters α and β if $W = \left(\frac{X^k}{\lambda^k - X^k}\right)$, which is the odd ratio of power function distribution with shape parameter k and scale parameter λ.

$$f_Y(y) = \frac{\beta^\alpha w^{\alpha-1} e^{-\beta w}}{\Gamma(\alpha)}; \alpha, \beta > 0, 0 \le x \le \infty$$

Proof:

The proof is very easy to see by using simple transformation method. Note that

$$\frac{dx}{dw} = \frac{\left(\lambda^k - x^k\right)^2}{k\lambda^k x^{k-1}}$$

$$f_W(w) = f_X(x)\left|\frac{dx}{dw}\right|$$

Substitute the pdf, $f_X(x)$ of GPLD in (23) to have

$$f_W(w) = \frac{k\lambda^k \beta^\alpha x^{\alpha k-1}}{\Gamma(\alpha)\left(\lambda^k - x^k\right)^{\alpha+1}} exp[-\beta y] \left|\frac{\left(\lambda^k - x^k\right)^2}{k\lambda^k x^{k-1}}\right|$$

The rest of the proof is easy to see

$$f_W(w) = \frac{\beta^\alpha w^{\alpha-1} e^{-\beta w}}{\Gamma(\alpha)}$$

3.2 GPLD hazard, reverse hazard and other useful functions

If X is a continuous random variable with cdf $F_X(x)$, and pdf $f_X(x)$, then the survival, hazard, reverse hazard, cumulative hazard and mean residual life functions are given by $S_X(x) = 1 - F_X(x)$, $h_X(x) = f_X(x)/S_X(x)$, $\tau_X(x) = f_X(x)/F_X(x)$, $H_X(x) = -ln[S_X(x)]$, and $\omega_X(x) = \int_x^\infty S_U(u)\,du / S_X(x)$ respectively. The survival, hazard, reverse hazard and the mean residual life functions of GPLD are respectively given by

$$S_X(x) = 1 - \frac{1}{\Gamma(\alpha)}\gamma\left[\beta\left(\frac{x^k}{\lambda^k - x^k}\right), \alpha\right]. \quad (26)$$

$$h_X(x) = \frac{k\lambda^k \beta^\alpha x^{\alpha k-1} exp\left[-\beta\left(\frac{x^k}{\lambda^k - x^k}\right)\right]}{\Gamma(\alpha)\left(\lambda^k - x^k\right)^{\alpha+1}\left\{1 - \frac{1}{\Gamma(\alpha)}\gamma\left[\beta\left(\frac{x^k}{\lambda^k - x^k}\right), \alpha\right]\right\}}. \quad (27)$$

$$\tau_X(x) = \frac{k\lambda^k \beta^\alpha x^{\alpha k-1} exp\left[-\beta\left(\frac{x^k}{\lambda^k - x^k}\right)\right]}{\Gamma(\alpha)\left(\lambda^k - x^k\right)^{\alpha+1}\gamma\left[\beta\left(\frac{x^k}{\lambda^k - x^k}\right), \alpha\right]} \quad (28)$$

$$H_X(x) = -ln\left\{1 - \frac{1}{\Gamma(\alpha)}\gamma\left[\beta\left(\frac{x^k}{\lambda^k - x^k}\right), \alpha\right]\right\} \quad (29)$$

$$\omega_X(x) = \frac{\int_x^\infty \left\{1 - \frac{1}{\Gamma(\alpha)}\gamma\left[\beta\left(\frac{u^k}{\lambda^k - u^k}\right), \alpha\right]\right\} du}{1 - \frac{1}{\Gamma(\alpha)}\gamma\left[\beta\left(\frac{x^k}{\lambda^k - x^k}\right), \alpha\right]} \quad (30)$$

The graph of hazard function of GPLD for some parameters values are given in Figure2. The plots depict various shapes, positively skewed, negatively skewed, leptokurtic, and platykurtic shapes. This makes GPLD very attractive with high degree of flexibility, which makes the hazard rate function useful and suitable for different types of data observed in real situations.

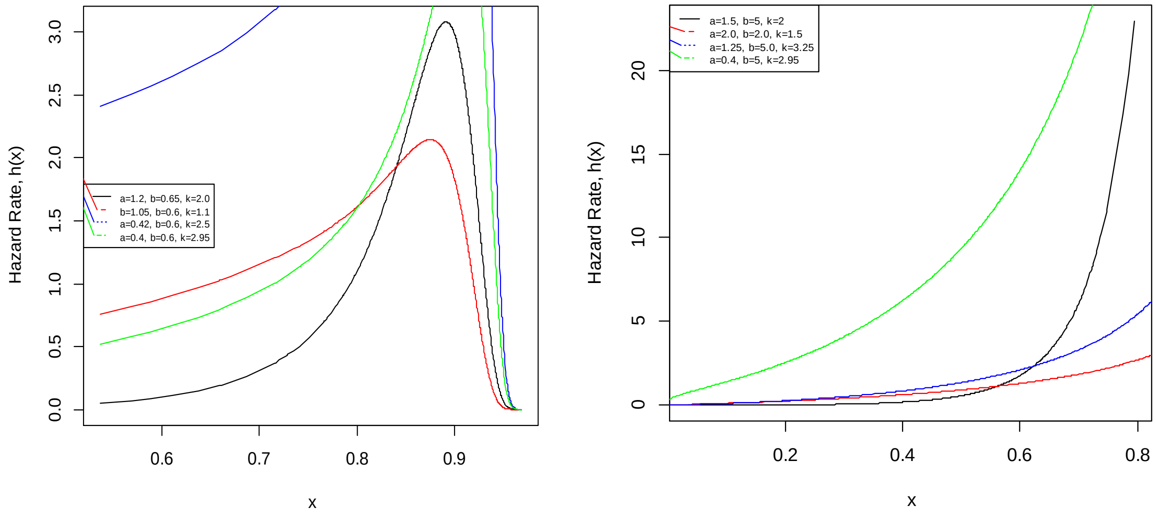

Figure 2: **Graph of GPLD Hazard Function for some Parameters Values for λ = 1**

3.3 GPLD quantile function

If T follows a gamma distribution, then random variate, X, can be generated via the simple transformation given by

$$X = \lambda\left(\frac{T}{1+T}\right)^{1/k}$$

where k is a shape parameter and λ is a scale parameter from the power function distribution. It is easy to generate T using R codes (R is a programming language, and should not be confused with R random variable). The *rgamma* generates random values of gamma distribution, T.

Theorem 3. If T follows a gamma distribution with parameters α and β, then the median of GPLD with parameters α, β, k and λ is given by

$$Q_X(0.5) = \lambda\left(\frac{Q_T(0.5)}{1+Q_T(0.5)}\right)^{1/k}$$

where $Q_T(0.5)$is the median of gamma distribution with parameters α and β.

Proof:

It is easy to see the result from Lemma 2(ii). Insert the median of gamma distribution into Lemma 2(ii) and we have

$$Q_X(0.5) = \lambda\left(\frac{Q_T(0.5)}{1+Q_T(0.5)}\right)^{1/k} \quad (31)$$

where $Q_T(0.5)$ is the median of gamma distribution with parameters α and β. Equation (31) completes the proof.

The quantile function is very useful in simulation studies and it is also used in deriving measures of partition, like the quartile, octile, decile and percentile. The 1st quartile of GPLD is given by

$$M = Q_X(0.25) = \lambda\left(\frac{Q_T(0.25)}{1+Q_T(0.25)}\right)^{1/k} \tag{32}$$

4 Moments, Moment Generating Function, Mean and Median Deviations

The moments, moment generating function, meanand median deviations for the GPLD are presented in this section. These measures are very useful and are obtained using the series expansion form of the pdf of GPLD.

4.1 Moments and Moment Generating Function of GPLD

The *r*th Moment

Theorem 4: Let X be a random variable that follows a GPLD(α, β, k, λ), the mean of X, is a weighted power function mean with parameter αk. and it is given by$E(X) = \frac{(\alpha-1)!\beta^{\alpha}\alpha k\lambda}{\Gamma(\alpha)(\alpha k+1)}$. where $\frac{\alpha k\lambda}{(\alpha k+1)}$ is the mean of power function distribution with shape parameter αk and scale parameter λ.

Proof:

We need to derive the *r*th raw moment first using the linear expansion of the GPLD pdf given in (24). Recall the pdf given in (24) as

$$f_X(x) = \frac{k\beta^{\alpha}}{\Gamma(\alpha)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^i(\alpha+i+j)!\,x^{k(\alpha+i+j)-1}}{i!\,j!\,\lambda^{k(\alpha+i+j)}}$$

By definition, the *r*th raw moment of GPLD is given by

$$E(X^r) = \frac{k\beta^{\alpha}}{\Gamma(\alpha)}\int_0^{\lambda} x^r\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^i(\alpha+i+j)!\,x^{k(\alpha+i+j)-1}}{i!\,j!\,\lambda^{k(\alpha+i+j)}}dx$$

$$= \frac{k\beta^{\alpha}}{\Gamma(\alpha)}\int_{x=0}^{\lambda}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^i(\alpha+i+j)!\,x^{k(\alpha+i+j)-1+r}}{i!\,j!\,\lambda^{k(\alpha+i+j)}}dx$$

It is clear that the integral of sum is equal to the sum of the integral, so, we have

$$= \frac{k\beta^{\alpha}}{\Gamma(\alpha)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\int_{x=0}^{\lambda}\frac{(-1)^i(\alpha+i+j)!\,x^{k(\alpha+i+j)-1+r}}{i!\,j!\,\lambda^{k(\alpha+i+j)}}dx$$

$$= \frac{k\beta^{\alpha}}{\Gamma(\alpha)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left[\frac{(-1)^i(\alpha+i+j)!\,x^{k(\alpha+i+j)+r}}{i!\,j!\,\lambda^{k(\alpha+i+j)}[k(\alpha+i+j)+r]}\right]_0^{\lambda}$$

Hence, the rth moment of GPLD is given by

$$E(X^r) = \frac{k\lambda^r\beta^\alpha}{\Gamma(\alpha)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^i(\alpha+i+j)!}{i!\,j!\,[k(\alpha+i+j)+r]} \quad (33)$$

The Mean of GPLD

If r = 1, we have the mean of GPLD given by

$$E(X) = \frac{k\lambda\beta^\alpha}{\Gamma(\alpha)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^i(\alpha+i+j)!}{i!\,j!\,[k(\alpha+i+j)+1]}$$

The mean can be easily readable if $i = j = 0$, the mean of GPLD becomes

$$E(X) = \frac{(\alpha-1)!\beta^\alpha\alpha k\lambda}{\Gamma(\alpha)(\alpha k+1)} \quad (34)$$

$$E(X) = \frac{(\alpha-1)!\beta^\alpha}{\Gamma(\alpha)}E(Z) \quad (35)$$

Thus, equation (35) completes the proof, where E(Z) is the mean of a two parameter power function distribution with shape parameter αk and scale parameter λ.

Moment Generating Function of GPLD

By definition, the moment generating function of a random variable X is given by

$$\mathrm{M}_X(t) = E(e^{tx}) = \int_{x=0}^{1} e^{tx} f_X(x)\,dx \quad (36)$$

Insert the pdf of GPLD in (24) into equation (36) to have

$$\mathrm{M}_X(t) = \frac{k\beta^\alpha}{\Gamma(\alpha)}\int_0^\lambda e^{tx}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^i(\alpha+i+j)!\,x^{k(\alpha+i+j)-1}}{i!\,j!\,\lambda^{k(\alpha+i+j)}}dx$$

$$= \frac{k\beta^\alpha}{\Gamma(\alpha)}\int_0^\lambda\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{tx}(-1)^i(\alpha+i+j)!\,x^{k(\alpha+i+j)-1}}{i!\,j!\,\lambda^{k(\alpha+i+j)}}dx$$

It is clear that the integral of sum is equal to the sum of the integral, so, we have

$$\mathrm{M}_X(t) = \frac{k\beta^\alpha}{\Gamma(\alpha)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\int_0^\lambda\frac{(-1)^i(\alpha+i+j)!\,x^{k(\alpha+i+j)-1}e^{tx}}{i!\,j!\,\lambda^{k(\alpha+i+j)}}dx$$

But we know that the series expansion of e^{tx} is given by

$$e^{tx} = \sum_{l=0}^{\infty}\frac{(tx)^l}{l!} = 1 + tx + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \cdots$$

$$\mathrm{M}_X(t)=\frac{k\beta^{\alpha}}{\Gamma(\alpha)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\int_{0}^{\lambda}\frac{(-1)^{i}(\alpha+i+j)!\,x^{k(\alpha+i+j)+l-1}t^{l}}{i!\,j!\,l!\,\lambda^{k(\alpha+i+j)}}dx$$

Hence, the moment generating function of GPLD is given by $\mathrm{M}_X(t)=\frac{k\beta^{\alpha}}{\Gamma(\alpha)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{i}(\alpha+i+j)!\lambda^{l}t^{l}}{i!j!l!\{k(\alpha+i+j)+l\}}$ (37)

$$\mathrm{M}_X'(t)=\frac{k\beta^{\alpha}}{\Gamma(\alpha)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{i}(\alpha+i+j)!\,\lambda^{l}lt^{l-1}}{i!\,j!\,l!\,\{k(\alpha+i+j)+l\}}$$

If l = 0, 1, 2,…, we have the series expansion of equation (37) to be

$$\mathrm{M}_X'(t)=0+\frac{k\beta^{\alpha}}{\Gamma(\alpha)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i}(\alpha+i+j)!\,\lambda}{i!\,j!\,\{k(\alpha+i+j)+1\}}+\frac{2k\beta^{\alpha}}{\Gamma(\alpha)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i}(\alpha+i+j)!\,\lambda^{2}t}{i!\,j!\,\{k(\alpha+i+j)+2\}}+\cdots$$

Set t = 0, we have the mean of GPLD as

$$\mu=E(X)=\mathrm{M}_X'(0)=\frac{k\lambda\beta^{\alpha}}{\Gamma(\alpha)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i}(\alpha+i+j)!}{i!j!\{k(\alpha+i+j)+1\}} \quad (38)$$

4.2 Mean and median deviations of GPLD

If X follows the GPLD distribution, by definition, the mean deviation about themean μ is given by

$$D_{\mu}=\int_{x=0}^{\lambda}|x-\mu|f_X(x)\,dx=2\mu f_X(\mu)-2\mu+2T(\mu),$$

and the median deviation about the median M is given by

$$D_{M}=\int_{x=0}^{\lambda}|x-M|f_X(x)\,dx=2T(M)-\mu,$$

where μ = E(X) is given in equation (34), M = $Q_X(0.5)$ in equation (32)and $\mathrm{T}(a)=\int_{a}^{\lambda}xf_X(x)\,dx$, then

$$f_X(x)=\frac{k\beta^{\alpha}}{\Gamma(\alpha)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i}(\alpha+i+j)!\,x^{k(\alpha+i+j)-1}}{i!\,j!\,\lambda^{k(\alpha+i+j)}}$$

$$D_{\mu}=\frac{2\mu k\beta^{\alpha}}{\Gamma(\alpha)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i}(\alpha+i+j)!\,\mu^{k(\alpha+i+j)-1}}{i!\,j!\,\lambda^{k(\alpha+i+j)}}-2\mu+2\int_{\mu}^{\lambda}\frac{k\beta^{\alpha}}{\Gamma(\alpha)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i}(\alpha+i+j)!\,x^{k(\alpha+i+j)}}{i!\,j!\,\lambda^{k(\alpha+i+j)}}dx$$

$$D_\mu = \frac{2\mu k\beta^\alpha}{\Gamma(\alpha)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^i(\alpha+i+j)!\,\mu^{k(\alpha+i+j)-1}}{i!\,j!\,\lambda^{k(\alpha+i+j)}} - 2\left\{\frac{k\beta^\alpha}{\Gamma(\alpha)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^i(\alpha+i+j)!\,\mu^{k(\alpha+i+j)+1}}{i!\,j!\,\lambda^{k(\alpha+i+j)}[k(\alpha+i+j)+1]}\right\}$$

$$D_\mu = 2\mu\frac{k\beta^\alpha}{\Gamma(\alpha)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^i(\alpha+i+j)!}{i!j!\lambda^{k(\alpha+i+j)}[k(\alpha+i+j)+1]}\left\{\mu^{k(\alpha+i+j)-1}[k(\alpha+i+j)+1] - \mu^{k(\alpha+i+j)}\right\} \quad (39)$$

The mean deviation can be easily readable if $j = i = 0$, the mean deviation of GPLD becomes

$$D_\mu = 2\mu\frac{k\beta^\alpha}{\Gamma(\alpha)}\frac{\alpha!\,\mu^{\alpha k}\{\mu^{-1}[\alpha k+1]-1\}}{\lambda^{\alpha k}[\alpha k+1]} \quad (40)$$

The median deviation is

$$D_M = 2T(M) - \mu,$$

$$D_M = 2\int_M^\lambda x\frac{k\beta^\alpha}{\Gamma(\alpha)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^i(\alpha+i+j)!\,x^{k(\alpha+i+j)}}{i!\,j!\,\lambda^{k(\alpha+i+j)}}dx - \mu,$$

$$D_M = 2\int_M^\lambda x\frac{k\beta^\alpha}{\Gamma(\alpha)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^i(\alpha+i+j)!\,x^{k(\alpha+i+j)}}{i!\,j!\,\lambda^{k(\alpha+i+j)}}dx - \mu,$$

$$D_M = 2\frac{k\beta^\alpha}{\Gamma(\alpha)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^i(\alpha+i+j)!}{i!\,j!\,\lambda^{k(\alpha+i+j)}k(\alpha+i+j)+1}\left\{\lambda^{k(\alpha+i+j)+1} - M^{k(\alpha+i+j)+1}\right\} - \mu$$

$$D_M = \frac{k\beta^\alpha}{\Gamma(\alpha)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^i(\alpha+i+j)!}{i!\,j!\,[k(\alpha+i+j)+1]}\left\{\frac{2\left(\lambda^{k(\alpha+i+j)+1} - M^{k(\alpha+i+j)+1}\right)}{\lambda^{k(\alpha+i+j)}} - \lambda\right\}$$

$$D_M = \frac{k\beta^\alpha}{\Gamma(\alpha)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^i(\alpha+i+j)!\left(\lambda^{k(\alpha+i+j)+1} - 2M^{k(\alpha+i+j)+1}\right)}{i!\,j!\,[k(\alpha+i+j)+1]\lambda^{k(\alpha+i+j)}} \quad (41)$$

The median deviation can be easily readable if $j = i = 0$, the mean deviation of GPLD becomes

$$D_M = \frac{k\beta^\alpha}{\Gamma(\alpha)}\frac{\alpha!\left(\lambda^{\alpha k+1} - 2M^{\alpha k+1}\right)}{[\alpha k+1]\lambda^{\alpha k}}, \quad (42)$$

where M is the median of GPLD given in (32)

5 Maximum Likelihood Estimation of GPLD

Let $x_1, x_2, ..., x_n$ be a random sample from the GPLD with pdf given in (23), then the likelihood function is given by

$$L(\alpha, \beta, k) = \frac{k^n \lambda^{kn} \beta^{\alpha n}}{[\Gamma(\alpha)]^n} \prod_{i=1}^{n} \frac{x_i^{\alpha k-1}}{\left(\lambda^k - x_i^k\right)^{\alpha+1}} exp\left[-\beta \sum_{i=1}^{n} \left(\frac{x_i^k}{\lambda^k - x_i^k}\right)\right] \quad (43)$$

Take the log of the likelihood in (48) to have the log-likelihood function of GPLD given by

$$l = nlnk + nkln\lambda + anln\beta - nln\Gamma(\alpha) + (\alpha k - 1)\sum_{i=1}^{n} lnx_i - (\alpha + 1)\sum_{i=1}^{n} ln\left(\lambda^k - x_i^k\right) - \beta \sum_{i=1}^{n}\left(\frac{x_i^k}{\lambda^k - x_i^k}\right) \quad (44)$$

The maximum likelihood estimation parameters of the GPLD are given by differentiating l partially with each of the parameters and equating the result to zero to solve for the parameters.

$$\frac{\partial l}{\partial \alpha} = -nln\beta + k\sum_{i=1}^{n} lnx_i - \sum_{i=1}^{n} \ln\left(\lambda^k - x_i^k\right).$$

$$\hat{\beta} = \exp\left[\frac{k}{n}\sum_{i=1}^{n} lnx_i - \frac{1}{n}\sum_{i=1}^{n} \ln\left(\lambda^k - x_i^k\right)\right]. \quad (45)$$

$$\frac{\partial l}{\partial \beta} = \frac{\alpha n}{\beta} - \sum_{i=1}^{n}\left(\frac{x_i^k}{\lambda^k - x_i^k}\right).$$

$$\hat{\alpha} = \frac{\hat{\beta}}{n}\sum_{i=1}^{n}\left(\frac{x_i^k}{\lambda^k - x_i^k}\right). \quad (46)$$

$$\frac{\partial l}{\partial k} = \frac{n}{k} + nln\lambda + \alpha\sum_{i=1}^{n} lnx_i - k(\alpha + 1)\sum_{x=0}^{n} \frac{x_i^{k-1}}{\left(\lambda^k - x_i^k\right)} - \beta\lambda^k k \sum_{i=1}^{n} \frac{x_i^{k-1}}{\left(\lambda^k - x_i^k\right)^2}. \quad (47)$$

The equations obtained by setting the partial derivatives l with respect to k to zero is not in closed form and the values of the parameters k must be found by using numerical methods. The maximum likelihood estimates of the parameter, denoted by $\hat{k}$would be estimated using Newton-Raphsonprocedure provided by R package (maxLik or optim). The upper bound parameter, λ, can be estimated from data as the maximum value of the data plus a small positive value less than 1. This is to ensure that $\lambda > \max(x)$, where $x \in X$ of the variable of interest in the dataset.

6 Simulation Study and Applications

In this section, simulation study and real life application of the GPLD were carried out to proof the consistency and the usefulness of the maximum likelihood GPLD parameter estimates.

6.1 Simulation study

The maximum likelihood method for estimating the performance of GPLD is evaluated using Monte Carlo simulation for a total of eighteen parameter combinations with 1000 replications. Three different sample sizes n= 20, 200 and 1000 are considered, for small, medium and large samples respectively. The actual values, maximum likelihood estimates and standard errors of the parameter estimates are presented in Table 2. From Table 2, it is noted that the maximum likelihood parameter estimates performed well for estimating the distribution parameters. As the sample size increases, the standard error decreases as expected.

Table 2: Actual values, Average Estimates and Standard error for various parameter values

	Actual values				Estimates				Std Error			
N	α	β	k	λ	$\hat{\alpha}$	$\hat{\beta}$	$\hat{k}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{k}$	$\hat{\lambda}$
20	0.5	0.5	1.0	5	0.39	0.42	1.04	3.70	0.1004	0.1570	0.5341	0.0650
	0.5	1.0	2.0	10	1.01	3.15	2.30	9.22	0.2823	1.1236	0.9736	0.8900
	1.0	0.5	1.0	5	1.24	0.65	1.03	5.47	0.3531	0.2272	0.5411	0.0113
	1.0	1.0	2.0	10	1.24	1.01	2.39	10.94	0.3537	0.3527	1.3898	0.0439
	1.5	0.5	1.0	5	1.41	0.65	0.90	5.31	0.4045	0.2225	0.3708	0.0049
	1.5	1.0	2.0	10	1.41	1.30	2.43	10.37	0.4045	0.4452	0.9399	0.0068
200	0.5	0.5	1.0	5	0.46	0.52	0.92	4.54	0.0381	0.0690	0.4097	0.0023
	0.5	1.0	2.0	10	0.46	0.95	0.92	11.51	0.0375	0.1256	0.8736	0.2123
	1.0	0.5	1.0	5	1.10	0.51	1.28	5.79	0.0822	0.0559	0.3411	0.0031
	1.0	1.0	2.0	10	1.06	1.01	2.04	11.33	0.0939	0.1142	0.6283	0.0089
	1.5	0.5	1.0	5	1.39	0.50	1.24	5.69	0.1255	0.0543	0.6630	0.0024
	1.5	1.0	2.0	10	1.39	0.99	2.28	11.02	0.1255	0.1085	0.9398	0.0052
1000	0.5	0.5	1.0	5	0.48	0.51	0.98	5.85	0.0178	0.0297	0.0861	0.0007
	0.5	1.0	2.0	10	0.51	1.04	1.86	11.60	0.0190	0.0603	0.4746	0.0026
	1.0	0.5	1.0	5	1.03	0.53	1.26	5.79	0.0408	0.0267	0.1965	0.0006
	1.0	1.0	2.0	10	1.00	1.04	2.02	11.53	0.0396	0.0525	0.4465	0.0023
	1.5	0.5	1.0	5	1.58	0.56	0.98	5.67	0.0646	0.0267	0.2075	0.0004
	1.5	1.0	2.0	10	1.58	1.11	2.20	11.00	0.0645	0.0535	0.9397	0.0010

6.2 Application of distribution

In this section, two applications to real data sets were provided to illustrate the uses and importance of the proposed Gamma-Power function distribution (GPLD). The distribution parameters are estimated by the method of maximum likelihood and five goodness-of-fit statistics are evaluated to compare the flexibility of the GPLD distribution with other competing distributions: Weibull-Power Cauchy distribution (WPC), Power Cauchy distribution (PC), gamma distribution and Power function distribution.

The goodness-of-fit tests, Akaike information criterion (AIC), Anderson-Darling statistic (A), Cramer-von Mises statistic (W) and Kolmogorov-Smirnov statistic (K-S) are computed to compare the fitted distributions to the datasets. See Chen and Balakrishnan (1995) for detailed information of A and W. Generally, the criteria for selection of best model among competing models to the fit the data of interest, is the model with the smallest values of these statistics. The required computations are carried out in the R-language (R Development Core Team, 2009).

Application 1: Breaking Strengths of 100 YarnData

The first real data set represents breaking strengths of 100 yarn (Duncan, 1974): 66, 117, 132, 111, 107, 85, 89, 79, 91, 97, 68, 63, 61, 86, 78, 96,93, 61, 62, 60, 95, 96, 88, 62, 65, 92, 137, 91, 84, 96, 97, 60, 65, 64, 67, 80, 64,104, 66, 84, 92, 86, 64, 132, 94, 99, 62, 61, 64, 67, 99, 85, 95, 89, 102, 100, 98, 97,104, 64, 61, 98, 99, 102, 91, 95, 111, 104, 97, 98, 102, 109, 88, 91, 103, 94, 75, 73,76, 70, 71, 78, 77, 77, 71, 72, 68, 64, 60, 68, 69, 62, 62, 87, 69, 62, 92, 60, 66, 98.

The data set are depicted in Figure 3 and shows that there is a gap in the histogram with positive skewness (0.4958) and kurtosis (2.7964). Table 3 displays the maximum likelihood estimates of the parameters with their corresponding standard errors in brackets. Table 3 shows all the parameters of the GPLD distribution and other competing distributions.

Table 3: MLE of Parameters and Standard Errors for Breaking strengths data

Distribution	Parameter Estimates			
GPLD	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{k}$
	0.3424	0.0705	138.1246	
	(0.0386)	(0.0144)	(0.00698)	1.9412 (0.1943)
WPC	$\hat{c}$	$\hat{\alpha}$	$\hat{\sigma}$	
	0.6964	21.2441	99.5837	
	(0.1517)	(5.6077)	(1.0215)	
PC	$\hat{\alpha}$	$\hat{\sigma}$		
	13.8617	98.6978		
	(1.2646)	(0.9599)		
GAMMA	$\hat{a}$	$\hat{\beta}$		
	22.2495	0.2654		
	(3.1220)	(0.0377)		
POWER	$\hat{\lambda}$	$\hat{k}$		
	138.1246	2.9728		
	(0.00698)	(0.2996)		

Table 4 clearly shows that the GPLD distribution provides the best fit to the breaking strengths data amongWPC, PC, Gamma and Power distributions. Its AIC, A, W and K-S approach zero faster than other competing distributions. Figures 4 also support the results in favour of the GPLD model.

Table 4: Goodness-of-fit Statistics and Criteria for Breaking strengths data

Distribution	AIC	A	W	K-S
GPLD	311.5056	0.3888	0.0750	0.0457
WPC	769.4093	0.4656	0.0792	0.0785
PC	770.0498	0.7381	0.1205	0.0870
GAMMA	860.2893	2.3244	0.3733	0.1233
POWER	950.5711	2.6500	0.4132	0.2221

Application 2: Number of Successive Failures of the Air Conditioning System of a Fleet of 213 Boeing 720 Jet Airplanes

The second real data set consists of 213 observations on the number of successive failures of the air conditioning system of a fleet of 13 Boeing 720 jet airplanes (Proschan, 1963):

50, 130, 487, 57, 102, 15, 14, 10, 57, 320, 261, 51, 44, 9, 254, 493, 33, 18, 209, 41, 58, 60, 48, 56, 87, 11, 102, 12, 5, 14, 14, 29, 37, 186, 29, 104, 7, 4, 72, 270, 283, 7, 61, 100, 61, 502, 220, 120, 141, 22, 603, 35, 98, 54, 100, 11, 181, 65, 49, 12, 239, 14, 18, 39, 3, 12, 5, 32, 9, 438, 43, 134, 184, 20, 386, 182, 71, 80, 188, 230, 152, 5, 36, 79, 59, 33, 246, 1, 79, 3, 27, 201, 84, 27, 156, 21, 16, 88, 130, 14, 118, 44, 15, 42, 106, 46, 230, 26, 59, 153, 104, 20, 206, 5, 66, 34, 29, 26, 35, 5, 82, 31, 118, 326, 12, 54, 36, 34, 18, 25, 120, 31, 22, 18, 216, 139, 67, 310, 3, 46, 210, 57, 76, 14, 111, 97, 62, 39, 30, 7, 44, 11, 63, 23, 22, 23, 14, 18, 13, 34, 16, 18, 130, 90, 163, 208, 1,24, 70, 16, 101, 52, 208, 95, 62, 11, 191, 14, 71

The skewness and kurtosis of the data are 2.2332 and 8.7353 respectively. The data is positively skewed and very peaked as depicted in Figure 4.

Table 5: Maximum likelihood estimates of parameters and standard errors for Boeing Data

Distribution	Parameter Estimates			
GPLD	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{k}$
	0.2306830 (0.018858)	0.0146661 (0.00256)	1.05150 (0.0500)	0.39008 (0.02915)
WPC	$\hat{c}$	$\hat{\alpha}$	$\hat{\sigma}$	
	3.2915 (1.5537)	0.3467 (0.1648)	22.6104 (12.5758)	
PC	$\hat{\alpha}$	$\hat{\sigma}$		
	1.1652 (0.0776)	48.8228 (4.6173)		
GAMMA	$\hat{a}$	$\hat{\beta}$		
	0.8985934 3 (0.08208)	0.01008404 (0.001200)		
POWER	$\hat{\lambda}$	$\hat{k}$		
	1.05150 (0.0503)	0.39008 (0.02915)		

The maximum likelihood estimates of the parameters of the fitted distributions with their corresponding standard errors in brackets are given in Table 5. All the parameters of the GPLD are significant at the 5% significance level. The GPLD provides a better fit to the yarn data than the WPC, PC, Gamma and Power function distributions as shown in Table 6.

Table 6: Goodness-of-Fit Statistics and Criteria for Boeing data

Distribution	AIC	A	W	K-S
GPLD	826.6376	0.4074	0.0467	0.0426
WPC	1962.4300	0.4187	0.0624	0.0450
PC	1973.1370	0.1377	0.9367	0.0585
GAMMA	1968.0810	1.2851	0.2283	0.0710
POWER	2071.5000	1.3237	0.9451	0.0592

Table 6 shows that GPLD AIC, A and W approach zero faster than that of others, and has the smallest K-S statistic value compared to the other models.

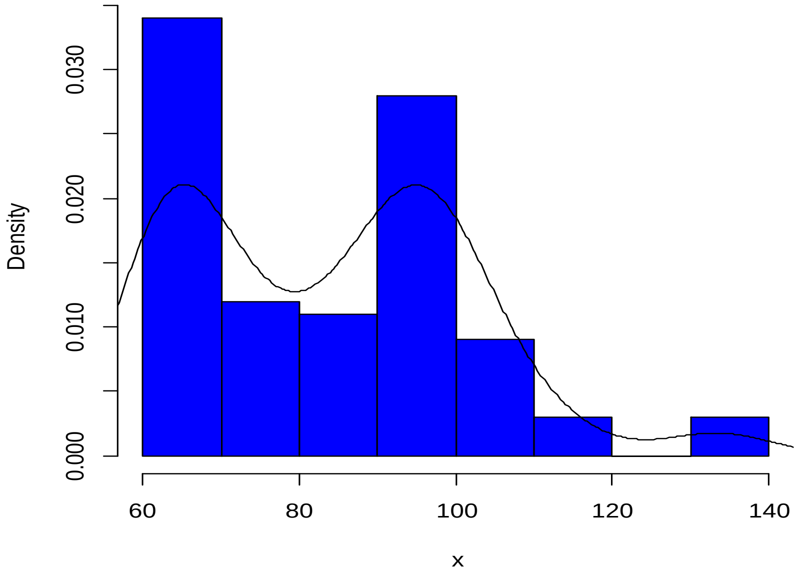

Figure 3: Breaking Strength of Yarn

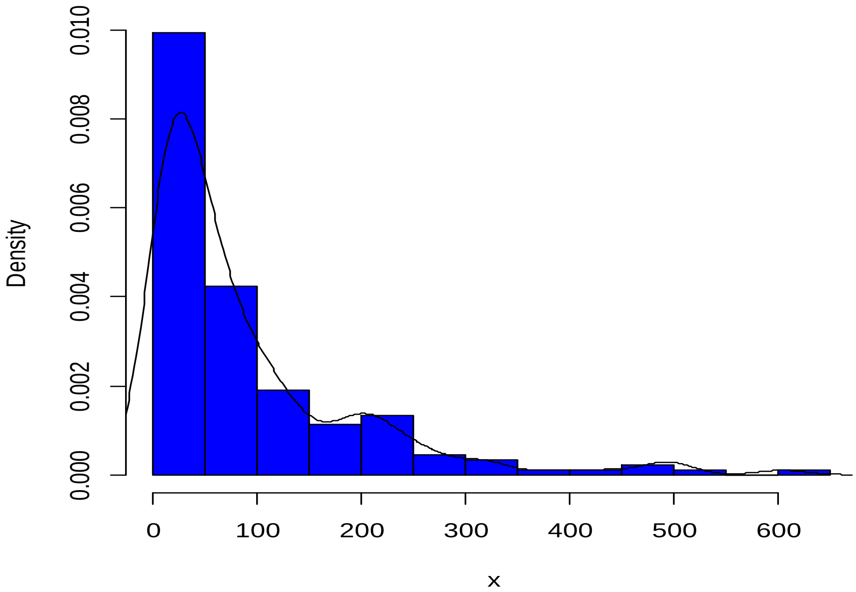

Figure 4: Boeing 720 Jet AirplaneData

7 Conclusion

This research proposed a new univariate continuous probability distribution called Gamma-Power function distribution with log-logistic quantile function (GPLD) using the *T*-R{Y} framework. The GPLD is a member of the *T*-Power{Y} family and results on its statistical properties are presented, such as the cumulative distribution function, density function, the quantile function, survival function, hazard function, cumulative hazard function, moments, and Shannon entropy. The maximum likelihood estimation of the parameters of the model were derived. GPLD distribution was applied to two data and the results of the GPLD outperform WPC, PC, Gamma and Power distributions. This is a clear indication that a convoluted distribution is a better model than its sub-models or distributions combined to form the convoluted distribution. The GPLD will be very useful to model data, where gamma or power function is not a good fit. It will also add to the body of existing continuous distributions.

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