On change of effective date of promotion in a university faculty system: a stochastic root-finding problem

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This study considers the problem of changing the effective date of promotion in a university faculty (system) as an embedding problem. The problem is whether the observed discrete-time transition matrix of the system can be expressed as a fractional matrix root in the stochastic sense. Conditions for embeddable stochastic matrices in the literature are adapted for the study. By the method based on the Cauchy's integral formula with the integrand defined on the Runnenburg's heart-shaped region, it is found that the transition matrix describing the system is not embeddable. The implications of changing the effective date of promotion in the system vis-à-vis the evolution of the expected personnel structure are highlighted.

Keywords: cauchy integral; embedding problem; manpower planning; markov chains; stochastic matrices.

1 Introduction

It is important to know beforehand whether certain managerial decisions could create bottlenecks for the planning division of a system so that possible trauma may be evaded. This study focuses on a system defined to be a faculty of a university setting in Nigeria. The study considers the methodological problem that may arise from a forward shift in the effective date of promotion. The task to extrapolate the personnel structure of the faculty is viewed as an embedding problem (Guerry, 2013; Ekhosuehi, 2021).

We revisit the embedding problem in discrete-time, albeit for a more practical situation encountered in a university faculty. Academics in the university system may be categorised according to the following hierarchical nomenclature: Graduate Assistant, Assistant Lecturer, Lecturer II, Lecturer I, Senior Lecturer, Associate Professor and Professor. These categories form the transient states of the university manpower system. The system also contains an absorbing state. The absorbing state includes all forms of wastage (or losses) due to resignations, termination of appointment, redundancy, ill-health, death, dismissal and retirement. Productivity of an individual academic staff in the university system is akin to the position the staff occupies in the academic reward structure defined by the grade levels/ranks. To attain the peak of the reward structure, the staff must work very hard to satisfy certain guidelines for appointments and promotions. Universities have the prerogative to frame guidelines for appointments and promotions in line with their visions and missions. The guidelines are a point-based system on a set of criteria which include: academic qualifications, publications and creative works, teaching/professional experience, conferences, administrative experience and general contribution. These guidelines set by the various institutional authorities influence the behaviour of most academics (Ayebelehin, 2021). The expectation of the individual staff in the system is to rise through the ranks to reach the peak of the career path of becoming a Professor at the earliest possible time. This is because this career peak is accorded a very high reputation and is significant with prestige. What is the fate of the 'fledgling' academic staff when the time period for promotion is extended? This study considers this question from a methodological perspective.

In the system considered herein, the university's calendar year begins from October 1 and ends on September 30 in the following year. Consequent upon this, promotion takes effect from October 1 in the staff performance evaluation year. There is a recent development in the system and the university is planning to harmonise the effective date of promotion with the national budgetary year starting from January 1 and ending at December 31 of the same year. By doing so, the effective date of promotion is extended by three months, from October 1 to January 1 of the following year. But there is a lack of data on this new unit time interval to extrapolate the staffing levels at the end of the forward shift in the interval. Suppose the change in the effective date of promotion is expected to be implemented in the 2021 promotion exercise. Then the academic planners may have to figure out a way to extrapolate the academic staff structure by an additional period of three months. This study is aimed at investigating the usability of the underlying extant homogeneous transition matrix describing the academic staff flows of a university faculty for this purpose. The study estimates the transition matrix of the first-order difference equation describing the system from available data and comments on the possible consequences of changing the effective date of promotion.

2 Literature Review

The personnel structure of a manpower system is usually extrapolated using a first-order difference equation wherein the transition dynamics of the system is defined by a transition matrix that satisfies the properties of a Markov chain (Bartholomew *et al.*, 1991). This approach has received considerable attention in the statistical aspects of manpower planning since the early work of Gani (1963) and it is still relevant in recent times (Nilakantan, 2015; Ekhosuehi, 2020; Esquível*et al.*, 2021).

In the first-order difference equation for a manpower system, the transition matrix is assumed to be homogeneous and the system is studied over equal time intervals or unit time intervals of length 1. The manpower system satisfying this description is said to be a Homogeneous Markov System (HMS). The HMS is a very special case of the Non-Homogeneous Markov Set System (NHMSS) defined in Vassiliou (2021).Extrapolation of the population structure of a HMS is straightforward when the unit time interval is of length 1, but this can pose computational challenges when the unit time interval is a fraction, say $1/p$, $p > 1$. There are instances where a transition matrix over a certain short time interval $1/p$ is required, but only the one-unit time interval Markov chain is available (Guerry, 2013; Ekhosuehi, 2021). When this is the case in the discrete-time context, the problem snowballs to that of finding the stochastic root of a stochastic matrix. This problem, which has gained prominence in the literature, is well-known as the embedding problem (Guerry, 2013, 2014, 2017, 2019; Ekhosuehi, 2021).

The embedding problem for discrete-time Markov chains has a long history. It dates back to the work of Leslie (1945). The problem is divided into two parts. The first part is the embedding problem of discrete-time Markov chains in a continuous-time one (Runnenburg, 1962; Johansen, 1974; Singer and Spilerman; 1974, 1976; Carette, 1995; Kreinin and Sidelnikova, 2001; Israel, Rosenthal and Wei, 2001; Davies, 2010; Ekhosuehi and Osagiede, 2013; Osagiede and Ekhosuehi, 2015; Esquível*et al.*, 2021) and the second part, which is more recent, is the embedding problem of discrete-time Markov chains in another discrete-time one (Guerry, 2013; 2014, 2017, 2019; Amenaghawon et al., 2020; Ekhosuehi, 2021). The work of Ekhosuehi (2021) generalises several results including that of Amenaghawon et al. (2020). A list of related literature to the embedding problem is presented in Table 1.

Higham and Lin (2011) had earlier studied the problem of finding the stochastic root of a stochastic matrix from the context of linear algebra. Guerry (2019) derived sufficiency conditions for a 3×3 stochastic matrix with real eigenvalues. The roots of a stochastic matrix may be studied using the concept of spectral decomposition and the properties of the matrix are examined for the existence stochastic roots (Guerry, 2021).Guerry (2021) examines matrix root properties and embedding conditions for discrete-time Markov chains with three states for a transition matrix having complex eigenvalues. Based on the spectral decomposition of the transition matrix, necessary and sufficient conditions for the existence of a $p-$th stochastic root of the transition matrix were presented. Embedding conditions derived from spectral decomposition demand that the transition matrix is diagonalisable. When the transition matrix is non-diagonalisable, then it needs to be perturbed by translating the transition matrix into a diagonalisable stochastic matrix with the eigenvalues arbitrarily close to the eigenvalues of the original matrix with the same principal eigenspaces. An algorithm to achieve this, which also preserves the spectral properties, is contained in Pauwelyn and Guerry (2020).

Table 1: Some contributors to the embedding problem of discrete-time Markov chains

S/N	Approach	Contributors
1.	Characterisation of eigenvalues within a heart-shaped region	Runnenburg (1962)
2.	Use of admissible logarithm of the Markov chain based on Sylvester's formula	Johansen (1974)
3.	Use of admissible logarithm of the Markov chain with regularisation algorithm	Singer and Spilerman (1974, 1976); Kreinin and Sidelnikova (2001); Israel, Rosenthal and Wei (2001); Davies (2010); Ekhosuehi and Osagiede (2013); Osagiede and Ekhosuehi (2015)
4.	Diagonalization transformation based on Jordan decomposition or spectral decomposition	Guerry (2013, 2014, 2017, 2019, 2021)
5.	Moment generating function (z-transform) approach	Amenaghawon, Ekhosuehi and Osagiede (2020)
6.	Use of Cauchy's integral formula	Ekhosuehi (2021)
7.	Calibration procedure and structural representation based on semi-Markov processes	Esquível, Krasii and Guerreiro (2021)

This study focuses on conditions, credited to Higham and Lin (2011), Guerry (2017) and Ekhosuehi (2021), for embeddable stochastic matrices in discrete-time. The study compares the performance of the existing conditions for identifying embeddable stochastic matrices. The comparison is important here in order to adjudge which of the conditions is most suitable for the system at hand. To validate the inference from the conditions in the literature for the embedding problem in discrete-time, this study relies on results from the MATLAB package (MATLAB R2007b). The choice of the MATLAB package is borne out of its usefulness as a symbolic programming software package and its ease in computing fractional roots of matrices.

3 Methodology

This section presents useful definitions, the primary source of data for the study, the general manpower setting and the conditions for embeddable stochastic matrices in discrete-time.

3.1 Preliminaries

To make our discussion clearer to a broader audience, the following definitions are provided.

Definition 1 A $k \times k$ real matrix $\mathbf{Q} = [q_{mn}]$ is called a stochastic matrix if $q_{mn} \geq 0$ for any $m, n = 1, 2, \cdots, k$ and $\sum_{n=1}^{k} q_{mn} = 1$ for any m.

Definition 2 (Ekhosuehi, 2021) The discrete-time Markov chain with transition matrix $\mathbf{Q}$ is said to be embedded in a process described by the matrix $\mathbf{A}$ if $f(\mathbf{Q}) = \mathbf{A}$, where f is a single-valued analytic function.

Definition 3 Let $\mathbb{R}$ denote the set of real numbers. The set defined by $\mathbb{C} = \{z : z = x + i\,y;\ x, y \in \mathbb{R},\ i^2 = -1\}$ is called the set of complex numbers.

Definition 4 For a $k \times k$ matrix $\mathbf{Q}$, the problem of finding $\lambda \in \mathbb{C}$ and $\mathbf{0} \neq \mathbf{x} \in \mathbb{C}^k$ such that $(\lambda \mathbf{I} - \mathbf{Q})\mathbf{x} = \mathbf{0}$, is called the eigenvalue-eigenvector problem.

The scalar $\lambda \in \mathbb{C}$ is called the eigenvalue of $\mathbf{Q}$ and $\mathbf{x} \in \mathbb{C}^k$ is called the eigenvector associated with λ. The spectrum of $\mathbf{Q}$ is

$$\text{spectrum}(\mathbf{Q}) = \{\lambda \in \mathbb{C} : \lambda \text{ is an eigenvalue of } \mathbf{Q}\}. \tag{3.1}$$

Definition 5 (Runnenburg, 1962) The heart-shaped region H_k for the spectrum of an embeddable stochastic matrix $\mathbf{Q} \in \mathbb{R}^{k \times k}$ is a complex plane whose boundary is the curve $x(v) + iy(v)$, where

$$x(v) = \left[\exp\left(-v + v\cos\frac{2\pi}{k}\right)\right]\cos\left(v\sin\frac{2\pi}{k}\right),\ y(v) = \left[\exp\left(-v + v\cos\frac{2\pi}{k}\right)\right]\sin\left(v\sin\frac{2\pi}{k}\right),$$

together with its symmetric image with respect to the real axis, $k =$ order of the matrix $\mathbf{Q}$ and v lies in the interval $0 \leq v \leq \dfrac{\pi}{\sin(2\pi/k)}$.

Definition 6 (Ekhosuehi, 2021) For a single-valued analytic function f defined within and on the heart-shaped region $\Gamma = H_k \subset \mathbb{C}$, the Cauchy's integral definition for the stochastic matrix $\mathbf{Q} \in \mathbb{R}^{k \times k}$ is

$$f(\mathbf{Q}) = \frac{1}{2\pi i}\int_{\Gamma} (\lambda \mathbf{I} - \mathbf{Q})^{-1} f(\lambda)\, d\lambda. \tag{3.2}$$

3.2 Data

We study the transition behaviour in the Faculty of Physical Sciences at University of Benin, Benin City. The Faculty was created from the then Faculty of Science in the

2004/2005 academic session. The university faculty system is viewed as a graded system through which members of staff move by recruitment, promotion and wastage. Promotion in the system follows a natural order and takes effect from October 1 in the year the staff member is appraised for promotion. Staff members are appraised every year according to the university guidelines for appointments and promotions. Data on the academic staff flows in the faculty are hierarchically structured. In a previous study, Ekhosuehi (2013) collated the data, which span the period 2005/2006 to 2011/2012 academic sessions. The data follows the flow pattern:

$$\sum_{n=1}^{7} x_{mn}(t) = x_{m,m}(t) + x_{m\,,\,m+1}(t), \qquad (3.3)$$

and

$$\sum_{n=1}^{7} x_{nm}(t) = 0, \text{ for } n > m, \qquad (3.4)$$

where $x_{mn}(t)$ is the academic staff flow from state m to state n at session t, $m,n = 1,2,\cdots,k$. Each session may be seen as a unit time interval. Equation (3.3) implies that there is no 'double promotion' during the period data are available and equation (3.4) shows that demotion is precluded in the system. Since equation (3.4) denotes demotion, the academic flow $x_{mn}(t)$, indexed by the natural numbers m,n, is reversed. The data in Ekhosuehi (2013) are utilised for this study. This is because the data provide useful information on the career path of the academic staff in the faculty.

3.3 The general setting

Consider the $k-$echelon manpower system, where the sequence, $\{S_m\}_{m=1}^{k}$, defines the states of the system with S_1 as the base level and S_k as the uppermost level. These states form a partition of the system, that is, $\#\left(\bigcup_{m=1}^{k}\{S_m\}\right) = k < \infty$ and $\#\left(\bigcap_{m=1}^{k}\{S_m\}\right) = 0$. The states may be ordered (Guerry, 2017). The population structure of the system is represented by the $k-$tuple row vector $\mathbf{x}(t) = [x_1(t), x_2(t), \cdots, x_k(t)]$ whose components are random variables with $x_m(t)$ being the number of members in state m at time t, $m = 1,2,\cdots,k$. Let the sequence, $\{X(t)\}_{t=0}^{\infty}$, be a known realisation of a known stochastic process. Let $\mathbf{P} = [p_{mn}]$ denote the homogeneous probability matrix with p_{mn} as the probability of a member of the system moving from state m to state n during the unit time interval, $m\,,n = 1,2,\cdots,k$. Let $\mathbf{w} = [w_1, w_2, \cdots, w_k]$ represent the wastage vector, wherein w_m is the probability of a member of the population leaving the system from state m during the unit time interval. Let $\mathbf{r} = [r_1, r_2, \cdots, r_k]$ be the recruitment vector with r_m being the probability of a new member entering the system in state m at the unit time interval. These transition parameters are usually estimated from historical data set of the system using the method of maximum likelihood (Bartholomew *et al.*, 1991).

The expected population structure, $E[\mathbf{x}(t+1)]$, is of interest. This is because the application of Markov chains to manpower systems (including, the university system) is treated from a deterministic perspective by most research (De Feyter, 2006). Following extant research guide (Bartholomew *et al.*, 1991), the time frame of reference is discrete and the population structure $E[\mathbf{x}(t+1)]$ is assumed to evolve according to the first-order difference equation:

$$E[\mathbf{x}(t+1)] = E[\mathbf{x}(t)]\mathbf{P} + R(t+1)\mathbf{r},\ t = 0,1,2,\cdots, \tag{3.5}$$

where $R(t+1)$ is the recruitment level at time $t+1$. The matrix $\mathbf{P}$ is sub-stochastic. The sub-stochastic form of $\mathbf{P}$ is attributed to wastage. Since wastage and recruitment are allowed, the Markov system is said to be open. Open Markov-type population models are interesting, important and worthy of further research (Esquível *et al.*, 2021).Let this open Markov system operates in such a manner that the recruitment comprises two groups of new entrants: those recruited to replace leavers and those needed to achieve expansion of the system. Then new memberships are created in the system according to the equation

$R(t+1) = \mathbf{x}(t)\mathbf{w}' + \Delta X(t+1), t = 0,1,2,\cdots,$(3.6)

where $\Delta X(t+1) = X(t+1) - X(t) > 0$. It follows obviously that $E[\mathbf{x}(t+1)]$ is now given by

$$E[\mathbf{x}(t+1)] = E[\mathbf{x}(t)]\left(\mathbf{P} + \mathbf{w}'\mathbf{r}\right) + \Delta X(t+1)\mathbf{r}, t = 0,1,2,\cdots. \tag{3.7}$$

Let the probability of movement of memberships within and between the system and the external environment be defined by a $k \times k$ matrix, $\mathbf{Q} = \mathbf{P} + \mathbf{w}'\mathbf{r}$. This matrix $\mathbf{Q} = [q_{mn}]$ is stochastic and is assumed to be stationary as is the case with HMS (Tsaklidis, 1994). Suppose there is a forward shift in the time interval by a fraction $\frac{1}{p}$, $p > 1$. Then

$$E[\mathbf{x}(t+1+1/p)] = E[\mathbf{x}(t+1)]\mathbf{Q}^{1+1/p} + \Delta X(t+1)\mathbf{r},\ t = 0,1,2,\cdots. \tag{3.8}$$

The time argument in $\Delta X(t+1)$ is unchanged because the expansion $\Delta X(t+1)$ is assumed invariant in the interval $[t, t+1]$. For $\mathbf{Q}^{1+1/p}$, we are faced with the problem of finding the stochastic root of the stochastic matrix $\mathbf{Q}$. That is, the problem of finding the $p-$th root of $\mathbf{Q}$ such that the result of $\mathbf{Q}^{1/p}$ is stochastic.

3.4 The governing conditions

The embedding problem of a discrete-time Markov chain with transition matrix $\mathbf{Q}$ may be reformulated in terms of the $p-$th root of $\mathbf{Q}$.There can be p solutions. For this reason, embedding conditions are formulated under which at least one of the matrix solutions to $\mathbf{Q}^{1/p}$ is stochastic. Studies on the embedding problem for a discrete-time Markov chain have involved a characterisation of the eigenvalues and the derivation of conditions under which a stochastic matrix is embeddable (Guerry, 2013, 2014, 2017, 2019, 2021; Amenaghawon et al., 2020; Ekhosuehi, 2021). Of particular interest in this study are the works of Higham and Lin (2011), Guerry (2017) and Ekhosuehi (2021). In these studies, conditions for a stochastic matrix to be embeddable are derived. In Higham and Lin (2011), it is stated that the necessary condition for a stochastic matrix $\mathbf{Q}$ to have stochastic roots is that it must satisfy the relation

$$|\mathbf{Q}| \le \prod_{m=1}^{k} q_{mm} . \tag{3.9}$$

By the method of matrix factorisation, Guerry (2017) showed that the stochastic root of the stochastic matrix $\mathbf{Q}$ exists for even roots if

$$tr(\mathbf{Q}) \ge 1. \tag{3.10}$$

These conditions are related to the eigenvalues of $\mathbf{Q}$ as $|\mathbf{Q}| = \prod_{m=1}^{k} \lambda_m$ and $tr(\mathbf{Q}) = \sum_{m=1}^{k} \lambda_m$. More recently, new conditions for embeddable stochastic matrices were derived based on Cauchy's integral definition. The conditions include (Ekhosuehi, 2021):

(i) $\lambda_r \in H_k \subset \mathbb{C}$ for all r;

(ii) $\sum_{m=1}^{k} g_{nm}^{r} = 0$ for each n, $r = 2,3,\cdots,k$;

(iii) $0 \le \sum_{r=1}^{k} f(\lambda_r) g_{nm}^{r} \le 1$ for each m, n,

where f is a root function and g_{mn}^{r} is defined to be $g_{mn}^{r} = \dfrac{\phi_{mn}(\lambda_r)}{\prod_{\substack{n=1 \\ n \ne r}}^{k} (\lambda_r - \lambda_n)}$,

with $\phi_{mn}(\lambda_r) = (-1)^{m+n} |M_{mn}(\lambda_r)|$, $M_{mn}(\lambda_r)$ being the $(k-1)\times(k-1)$ sub-matrix of $(\lambda\mathbf{I} - \mathbf{Q})$ constructed by deleting the $m-$th row and $n-$th column of $(\lambda\mathbf{I} - \mathbf{Q})$ and the determinant $|M_{mn}(\lambda_r)|$ is called the minor of the element in the intersection of the $m-$th row and $n-$th column of $(\lambda\mathbf{I} - \mathbf{Q})$.The swap of the subscripts indexing the element g_{mn}^{r} in (ii) and (iii) is as a result of matrix transposition when finding the inverse of $(\lambda\mathbf{I} - \mathbf{Q})$ in equation (3.2).

For the problem at hand, $k = 7$. To use the data in Ekhosuehi (2013), the 2011/2012 session is taken as the base session. For convenience so as not to clutter notations, the sessions are represented by the year-ending. That is, the 2011/2012 session is represented as 2012 and so on. The subsequent sessions are recoded as $2013 \equiv t = 1$, $2014 \equiv t = 2$, $2015 \equiv t = 3$, $\cdots$, $2020 \equiv t = 7$, $2021 \equiv t = 8 + 1/4$. The session code for 2021 took into cognisance the three-months extension in the unit time interval. As mentioned earlier, a homogeneous Markov system (HMS) is considered. In such a system, the transition probabilities are assumed to be stationary so that the transition matrix describing the system can be used to extrapolate the structure of the system. Since the transition matrix is assumed to be stationary, it is needless to update the data as in Ekhosuehi (2013) to the current session. This study utilises the aforementioned conditions of Higham and Lin (2011), Guerry (2017) and Ekhosuehi (2021)to assess whether it is possible to use the extant transition matrix of the faculty based on the historical dataset when the time unit of the Markov chain is extended and also compares the inferences from the conditions. To validate the inference from the conditions for the embedding problem in discrete-time, this study relies on results from the spectral decomposition of the observable transition matrix to reach a final decision. Spectral decompositions of square matrices, that are not necessarily stochastic matrices, are easily obtained in the MATLAB environment. This is achieved by using the syntax `[V,D]=eig(Q)` to compute the matrix `V` with the $m-$th column being the eigenvector corresponding to the $m-$th diagonal entry in `D` of eigenvalues.

4 Results and Discussion

We estimate the entries of the transition probability matrix for the faculty as at 2012 by the method of maximum likelihood (Bartholomew *et al.*, 1991). The results of the transition parameters are presented in Table 2. The numbers 1 to 7 represent the hierarchical nomenclature of the system in the following order: Graduate Assistant, Assistant Lecturer, Lecturer II, Lecturer I, Senior Lecturer, Associate Professor and Professor. The upper off-diagonal entries of $\mathbf{P}$ give information on the promotion probabilities to the next higher level and the diagonal entries provide information on those that remain on the same grade level. Using the information contained in Table 2, we construct the stochastic matrix $\mathbf{Q}$, which is given in Table 3.

Table 2: Transition parameters of the university faculty

P	1	2	3	4	5	6	7	**w**
1	0.7084	0.2708	0	0	0	0	0	0.0208
2	0	0.6154	0.3385	0	0	0	0	0.0461
3	0	0	0.6882	0.3118	0	0	0	0
4	0	0	0	0.7872	0.1986	0	0	0.0142
5	0	0	0	0	0.8627	0.1242	0	0.0131
6	0	0	0	0	0	0.6341	0.3415	0.0244
7	0	0	0	0	0	0	0.8947	0.1053
r	0.5738	0.2787	0.0656	0.0328	0.0328	0	0.0164	

Table 3: The transition probability matrix of the faculty

Q	1	2	3	4	5	6	7
1	0.7203	0.2766	0.0014	0.0007	0.0007	0	0.0003
2	0.0265	0.6282	0.3415	0.0015	0.0015	0	0.0008
3	0	0	0.6882	0.3118	0	0	0
4	0.0081	0.0040	0.0009	0.7877	0.1990	0	0.0002
5	0.0075	0.0036	0.0009	0.0004	0.8632	0.1242	0.0002
6	0.0140	0.0068	0.0016	0.0008	0.0008	0.6341	0.3419
7	0.0604	0.0293	0.0069	0.0035	0.0035	0	0.8965

The matrix, $\mathbf{Q}$, provides information on the internal transitions in the system including the probability that losses in the system would result in a consequential recruitment to the system. The transition matrix of the system as at 2021 is the $(8+1/4)-$step transition matrix $\mathbf{Q}^{8+1/4}$. Now $\mathbf{Q}^{8+1/4}=\mathbf{Q}^{8}\mathbf{Q}^{1/4}$. Computing $\mathbf{Q}^{8}$ is straightforward. The challenge is to find $\mathbf{Q}^{1/4}$. To draw conclusions on whether the matrix $\mathbf{Q}$ is valid to extrapolate the personnel structure for the $1/4$ shift in the time interval, we characterise the eigenvalues of $\mathbf{Q}$ and examine them within the conditions for embeddable stochastic matrices. Put simply, we examine whether or not $\mathbf{Q}$ is embeddable.

We compute the eigenvalues of $\mathbf{Q}$ by solving eigenvalue-eigenvector problem $(\lambda\mathbf{I}-\mathbf{Q})\mathbf{x}=\mathbf{0}$ and then calculate their proportion by magnitude as well as the cumulative sum. The results are given in Table 4. From Table 4, it is easy to see that the spectral radius of $\mathbf{Q}$ is $\rho(\mathbf{Q})=1$ and 1 is a simple eigenvalue of $\mathbf{Q}$. Thus, $\mathbf{Q}$ is irreducible. Since a system with seven states is considered, the sufficiency conditions in Guerry (2019, 2021) are not applicable. The conditions in Guerry (2019, 2021) are developed for a three state system. Using the conditions in (3.9) and (3.10) for the existence of a stochastic root, we obtain: $|\mathbf{Q}|-\prod_{m=1}^{7}q_{mm}=0.002$ and $tr(\mathbf{Q})=5.2182$. These results suggest that the transition matrix $\mathbf{Q}$ is embeddable.

$z_1=0.8796+0.1478i$, $z_2=0.6856+0.1648i$, ` $z_3=0.5440+0.0586i$, $\bar{z}_j$ is the conjugate of z_j, $j=1,2,3$.

Table 4: Eigenvalues of the transition matrix

Eigenvalue	1	z_1	$\bar{z}_1$	z_2	$\bar{z}_2$	z_3	$\bar{z}_3$
Proportion	0.1891	0.1687	0.1687	0.1333	0.1333	0.1035	0.1035
Cumulative	0.1891	0.3578	0.5264	0.6597	0.7931	0.8965	1.0000

Next we examine the matrix $\mathbf{Q}$ using the conditions prescribed in Ekhosuehi (2021). First, we check whether the eigenvalues are contained in the heart-shaped region (i.e., $\lambda_r\in H_k\subset\mathbb{C}$ for all $r=1,2,\cdots,7$). The schematic representation of the heart-shaped region is shown in Figure 1. In Figure 1, two of the eigenvalues do not lie in the heart-shaped region, z_1 and its conjugate, $\bar{z}_1$. Thus, the matrix $\mathbf{Q}$ violates the first condition in Ekhosuehi (2021) and there is no need to consider the remaining two conditions therein. Since condition (i) is not satisfied, the transition matrix $\mathbf{Q}$ describing the system cannot be embeddable within the context of discrete-time Markov chains.

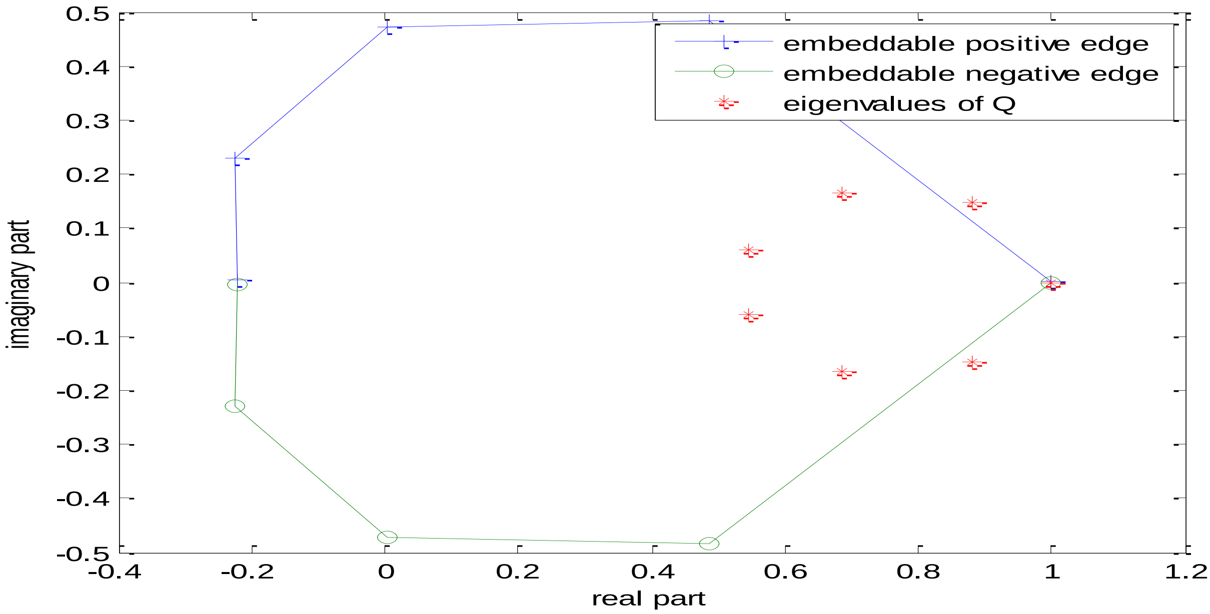

Figure 1: The Heart-Shaped Region Enclosing some Eigenvalues of the 7× 7 Stochastic Matrix, Q

There is now a conflict in reaching a decision. The first two approaches suggest that the transition matrix **Q** is embeddable, whereas the third approach due to Ekhosuehi (2021) suggests that **Q** is not embeddable. Since the embedding problem is a mathematical problem, we decide to evaluate the root $\mathbf{Q}^{1/4}$ using the MATLAB package (MATLAB R2007b). The MATLAB results are based on the spectral decomposition of matrices. The results are given in Table 5. Table 5 shows that the fractional matrix roots of **Q** exist, but the result is not a stochastic root as it contains negative entries. It follows that the fractional matrix roots of **Q** is not valid in the sense of Markov chains. Hence, the transition matrix **Q** describing the system is not embeddable within the context of discrete-time Markov chains. We conclude that the necessary conditions according to Higham and Lin (2011) and Guerry (2017) may be satisfied for certain stochastic matrices, but such stochastic matrices may not be embeddable.

Furthermore, we investigate whether it is possible to regularise the transition matrix **Q** by finding an approximate stochastic matrix that is embeddable based on the 80% threshold on eigenvalues (Parmet et al., 2010). From Table 4, the cumulative proportion of the remaining eigenvalues when 0.8796 + 0.1478i and 0.8796 - 0.1478i are deleted is less than 80%. It follows that the transition matrix **Q** cannot be regularised.

Since **Q** is not embeddable and cannot be regularised, it follows that whenever the unit time interval of the system is changed to any other time, the stochastic root-finding problem is encountered. Thus, the management of the system may have to jettison the use of **Q** as the underlying transition matrix for extrapolating the personnel structure. By doing so, the management may have to go through the rigour of data collection on the transition variables to re-estimate the transition matrix for the system. In this sense, this study serves as an early-warning signal to the university management when considering an extension of the effective date of promotion from October 1 to January 1of the following year.

Table 5: Results for the 1/4-step transition probability matrix using MATLAB

```
Q^1/4 =
  Columns 1through3
   0.9199 + 0.0000i   0.0934 + 0.0000i  -0.0171
   0.0090 + 0.0000i   0.8889 + 0.0000i   0.1173 - 0.0000i
  -0.0004 + 0.0000i  -0.0001 + 0.0000i   0.9108 - 0.0000i
   0.0023 - 0.0000i   0.0008 + 0.0000i   0.0002 + 0.0000i
   0.0022 + 0.0000i   0.0008 - 0.0000i   0.0001 - 0.0000i
   0.0014 - 0.0000i   0.0007 - 0.0000i   0.0001 + 0.0000i
   0.0177 + 0.0000i   0.0064 + 0.0000i   0.0012 + 0.0000i
  Columns 4through6
   0.0043 - 0.0000i  -0.0006 - 0.0000i   0.0001 - 0.0000i
  -0.0183 - 0.0000i   0.0031 - 0.0000i  -0.0004 - 0.0000i
   0.0980 + 0.0000i  -0.0090 - 0.0000i   0.0010 - 0.0000i
   0.9421 - 0.0000i   0.0575 - 0.0000i  -0.0038 + 0.0000i
   0.0001 + 0.0000i   0.9639 - 0.0000i   0.0388 - 0.0000i
   0.0001 + 0.0000i   0.0001 - 0.0000i   0.8924
   0.0008 + 0.0000i   0.0009 - 0.0000i  -0.0001 - 0.0000i
  Column 7
   0.0000 + 0.0000i
   0.0003 + 0.0000i
  -0.0003 + 0.0000i
   0.0009 + 0.0000i
  -0.0059 + 0.0000i
   0.1051 + 0.0000i
   0.9731 + 0.0000i
```

5 Conclusion

The embedding problem has found application in the decision to change the effective date of promotion by a few months in a university faculty system. This study shows that the embedding problem is encountered when the effective date of promotion is extended. Conditions for embeddable stochastic matrices in the literature were compared. The findings indicate that the conditions according to Higham and Lin (2011) and Guerry (2017) are only necessary for the existence of stochastic matrix roots of the system, but the conditions due to Ekhosuehi (2021) are sufficient for the system at hand. Accordingly, it was found that the stochastic matrix describing a university faculty system does not have a stochastic root. The implications of changing the effective date of promotion are that: (i) the existing transition matrix is no longer valid to extrapolate the personnel structure of the system and (ii) new data need to be collected on the transition variables to re-estimate the transition matrix for the system. Based on this methodological consideration and the cost of data collection, we recommend that the university management should maintain the status quo of October 1 as the effective date of promotion. Further considerations on changing the effective date of promotion may involve a redesign of the university manpower system as a Non-Homogeneous Markov Set System (NHMSS), wherein the transition parameters do not only vary with time, but also are defined within probability confidence intervals. This is a worthwhile venture for future research.

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