On Beta-Akash distribution with applications

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In this paper, a Beta-Akash distribution that extends the Akash distribution has been introduced. Expansions for the cumulative distribution and probability density functions of the new distribution are given. Various properties of the new distribution such as hazard function, moments, cumulants, skewness, kurtosis, mean deviations, Bonferroni and Lorenz curves, Rényi and Tsallis entropies, and stress-strength reliability are discussed. Moment generating function and characteristic function of the new model were derived. Distribution and the moment of order statistic have been derived. The method of maximum likelihood was used for estimation of parameters. The new model is quite flexible in analysing positively skewed data. A real life data was used to demonstrate the flexibility of the new distribution.

Keywords: Akash distribution; hazard rate function; maximum likelihood estimation; stress-strength reliability; order statistics

1. Introduction

In many fields of endeavour, modelling and analysis of data collected from well-designed experiments or surveys are crucial. In particular, there is often the need to analyse data collected from the fields of medicine, engineering, economics, among others. Analysis of data from any of the aforementioned fields is often carried out using statistical techniques. To employ parametric statistical techniques, it is a common practice to assume that the data being analysed follow certain probability distributions and hence, the quality of methods used in statistical modelling depends heavily on the distribution(s) fitted to the data under consideration. Consequently, several studies focused on the development of standard probability distributions together with their statistical techniques. It may be noted that a distribution may be appropriate for a particular data in study while in a different study that same distribution may appear inadequate (Enogwe, *et al*., 2020). It also pertinent to note that most of the existing are inadequate for a lot of real-life data and for this reason, new distributions are always developed.

One of the new probability distributions used for modelling lifetime data is the Akash distribution. The Akash distribution was developed by Shanker (2015) as a two-component mixture of exponential distribution having scale parameter, θ, and a gamma distribution having shape parameter, 3 and scale parameter, θ, with mixing proportions, $p=\left(\theta^2/\theta^2+2\right)$ and $q=\left(2/\theta^2+2\right)$, such that $p+q=1$. Suppose X denote a random variable having the Akash distribution with parameter, θ, then the cumulative distribution function (cdf) and the corresponding probability density function (pdf) of the Akash distribution are, respectively, given by

$$F_A(x;\theta)=1-\left[1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right]e^{-\theta x};x>0,\theta>0 \qquad (1)$$

and

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$$f_A(x;\theta)=\frac{\theta^3}{\theta^2+2}\left(1+x^2\right)e^{-\theta x};x>0,\theta>0 \quad (2)$$

Shanker and Shukla (2017) also investigated the properties of the Akash distribution and demonstrated its superiority over the exponential and Lindley distributions using two real lifetime data from the fields of medicine and engineering. Another related study by Shanker (2018) used the Akash distribution to model sixteen different datasets to substantiate the claim that this distribution outperforms the exponential, Lindley and Shanker distributions respectively in modelling real-life phenomena.

Due to the fact that the Akash distribution depends on one scale parameter, it cannot be flexible in modelling real life data with varieties of tails. To make the Akash distribution more flexible for statistical modelling, several extensions of the Akash distributions suffice. Notable among them is the quasi-Akash distribution due to Shanker (2016), power Akash distribution introduced by Shanker and Shukla (2017), two-parameter Akash distribution due to Shanker and Shukla (2017), generalized Akash distribution due to Shanker*et al.*(2018), exponentiated Akash distribution by Okereke and Uwaeme (2018) and Transmuted Akash distribution by Almawajdeh (2019), inverse power Akash distribution by Enogwe *et al.* (2021).

A very interesting family of distribution which includes almost all well-known distributions as special cases is the beta-generated class of distributions introduced by Eugene *et al.* (2002). Several beta-generated class of distributions available in the literature include beta normal due to Eugene *et al.* (2002), Nadarajah and Gupta (2004), Famoye *et al.* (2005), Nadarajah and Kotz (2004), Nadarajah and Kotz (2006), Gupta and Nadarajah (2006), Akinsete *et al.* (2008), Akinsete *et al.* (2009), Souza *et al.* (2010), Amusan (2010), Cordeiro and Lemonte (2011a), Cordeiro and Lemonte (2011b), Castellares *et al.* (2011), Cordeiro *et al.* (2012), Jafari and Mahmoudi (2014), Shittu and Adepoju (2012), Alshawarbeh *et al.* (2012), Domma and Condino (2013), Cordeiro *et al.* (2013), Gomes *et al.* (2013), MirMostafaee *et al.* (2015), Ownuk (2015), Rodrigues *et al.* (2015), Dias *et al.* (2018), Fischer and Vaughan (2016), Merovci *et al.* (2016), among others.

The aim of this paper is to propose and study the beta Akash (BA) distribution. The motivation for introducing the distribution is that it allows for greater flexibility of tails and can be widely applied in many areas such as engineering, biology, insurance and medicine, among others. Another benefit of the new distribution is its ability of fitting skewed data that cannot be properly fitted by the baseline distribution.

According to Eugene *et al.* (2002), the cdf of beta-generated distribution is given by

$$F(x)=I_{G(x)}(a,b)=\frac{1}{B(a,b)}\int_0^{G(x)}t^{a-1}\left(1-t^{b-1}\right)dt=\frac{B(G(x);a,b)}{B(a,b)} \quad (3)$$

where $G(x)$ denotes the baseline distribution, $a>0$ and $b>0$ are shape parameters that controls skewness and vary tail weights of the generated model, $B(a,b)=\int_0^1 t^{a-1}\left(1-t^{a-1}\right)dt$ is the beta function and $I_{G(x)}(a,b)=B(G(x);a,b)=\int_0^{G(x)}t^{a-1}\left(1-t^{b-1}\right)dt$ is the incomplete beta function.

The corresponding pdf of the Beta-G distribution is given by

$$f(x)=\frac{g(x)}{B(a,b)}[G(x)]^{a-1}[1-G(x)]^{b-1} \quad (4)$$

where $g(x)=dG(x)/dx$ is the pdf of the baseline distribution.

The rest of the paper is organized as follow. In Section 2, we define the pdf, cdf and hazard function, the Beta Akash (BA) distribution. In Section 3, we study some properties of the proposed distribution including moments and their related measures, entropies, Bonferroni and Lorenz curves, distribution of order statistics for the BA distribution. Also, in Section 3, we obtain the maximum likelihood estimates (MLEs) of the model parameters and the corresponding asymptotic confidence intervals. In Section 5, we provide some application using real life data to illustrate the usefulness of the distribution and its sub-models. The paper is concluded in Section 5.

2. Beta Akash distribution

In this section, we introduce the three-parameter BA distribution by taking $G(x)$in Equation (1) to be the cdf of the Akash distributed random variable X. By using (1) in (3), the cdf the BA distribution takes the form

$$F_{BA}(x;\theta,a,b)=\frac{1}{B(a,b)}\int_0^{\left(1-\left(1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right)e^{-\theta x}\right)} t^{a-1}(1-t)^{b-1}\,dt = I_{\left(1-\left(1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right)e^{-\theta x}\right)}(a,b) \qquad (5)$$

for $x>0$, $\theta>0$, $a>0$, $b>0$. By putting (1) and (2) into (4), the pdf of the BA distribution becomes

$$f_{BA}(x)=\frac{\theta^3(1+x^2)e^{-\theta b x}}{B(a,b)(\theta^2+2)}\left(1-\left(1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right)e^{-\theta x}\right)^{a-1}\left(1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right)^{b-1} \qquad (6)$$

for $x>0$, $\theta>0$, $a>0$, $b>0$. Henceforth, a random variable, say, X, that follows the Beta Akash distribution will be denoted by $X\sim BA(\theta,a,b)$.
The hazard rate function for $X\sim BA(\theta,a,b)$ is given by

$$h_{BA}(x)=\frac{f_{BA}(x)}{1-F_{BA}(x)}=\frac{\frac{\theta^3(1+x^2)e^{-\theta b x}}{\theta^2+2}\left(1-\left(1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right)e^{-\theta x}\right)^{a-1}\left(1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right)^{b-1}}{B(a,b)\left[1-I_{\left(1-\left(1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right)e^{-\theta x}\right)}(a,b)\right]}. \qquad (7)$$

2.1 Shapes of the BA probability density function

Taking the natural log of (6), we obtain

$$\begin{aligned}\ln f_{BA}(x)=\ln\left(\frac{\theta^3}{B(a,b)(\theta^2+2)}\right)+(a-1)\ln\left(1-\left(1+\frac{\theta x(\theta x+2)}{\theta^3+2}\right)e^{-\theta x}\right)\\ +(b-1)\ln\left(1+\frac{\theta x(\theta x+2)}{\theta^3+2}\right)+\ln(1+x^2)-\theta b x\end{aligned} \qquad (8)$$

Differentiating (8) partially with respect to x, we get

$$\Rightarrow\frac{\partial}{\partial x}\log f_{BA}(x)=\frac{(a-1)\theta^3(1+x^2)e^{-\theta x}}{(\theta^2+2)F_A(x)}+\frac{2(b-1)\theta(\theta x+1)}{\theta^2+2+\theta x(\theta x+2)}+\frac{2x}{1+x^2}-b\theta \qquad (9)$$

where $F_A(x)$ is given in (1). The following can be deduced from (9):

(i) when $0 < a < 1$ or $a = 1$ and $\theta \geq 1$, then the pdf of the BA distribution is monotonically decreasing.
(ii) when $a > 1$ or $a = 1$ and $0 < \theta < 1$, then the pdf of the BA distribution is unimodal (increasing-decreasing) and attains its maximum at $x = x_0$, where

$$\left.\frac{\partial}{\partial x}\ln f_{BA}(x)\right|_{x=x_0} = 0 \qquad (10)$$

The plot of $f_{BA}(x)$ is given in Figure1 for some parameter values.

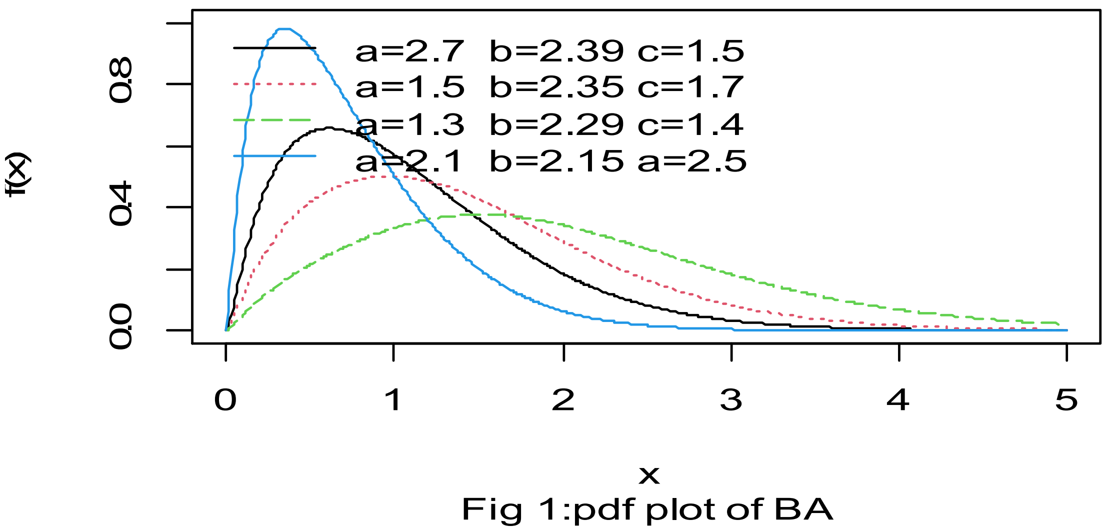

Figure 1: PDF of the BA Distribution for Varying Values of the Parameters α, θ, a and b.

2.2 Shapes of the hazard function of the BA distribution

We may define the log of the hazard rate function as

$$\ln h_{BA}(x) = \ln f_{BA}(x) - \ln(1 - F_{BA}(x)) \qquad (11)$$

The derivative of (11) with respect to x

$$\frac{\partial}{\partial x}\ln h_{BA}(x) = \frac{\partial}{\partial x}\ln f_{BA}(x) + h_{BA}(x) \qquad (12)$$

Substituting (7) and (8) into (12) gives

$$\frac{\partial}{\partial x}\ln h_{BA}(x) = \frac{(a-1)\theta^3(1+x^2)e^{-\theta x}}{(\theta^2+2)F_A(x)} + \frac{2(b-1)\theta(\theta x+1)}{\theta^2+2+\theta x(\theta x+2)} + \frac{2x}{1+x^2} - b\theta + \frac{\frac{\theta^3(1+x^2)e^{-\theta b x}}{\theta^2+2}\left(1-\left(1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right)e^{-\theta x}\right)^{a-1}\left(1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right)^{b-1}}{B(a,b)\left[1-I_{\left(1-\left(1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right)e^{-\theta x}\right)}(a,b)\right]} \qquad (13)$$

The following can be deduced from (13):

(i) If $0 < a < 1$, then the hazard rate function of the BA distribution is decreasing-increasing (bathtub shaped) and attains its minimum at $x = x_0$, where

$$\left.\frac{\partial}{\partial x} \ln h_{BA}(x)\right|_{x=x_0} = 0 \tag{14}$$

(ii) when $a \geq 1$, then the hazard rate function of the BA distribution is monotonically increasing. Figure 2 plots $h_{BA}(x)$ for some parameter values.

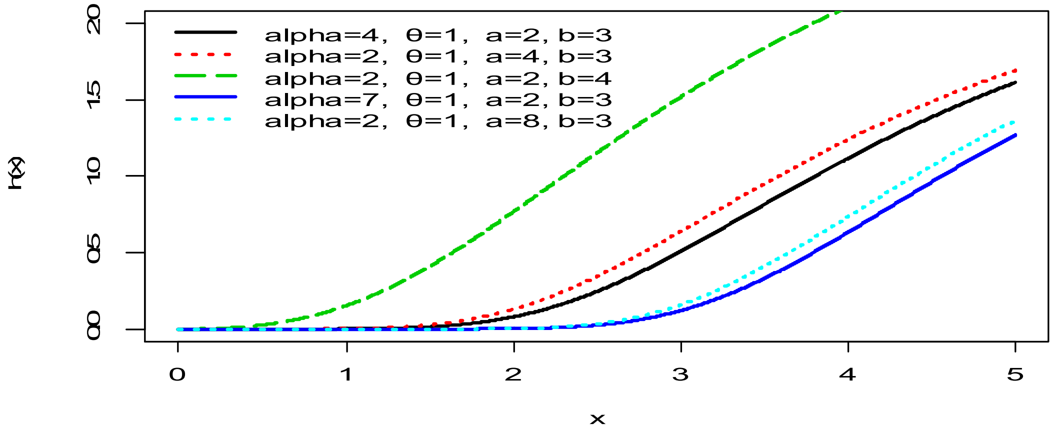

Figure 2. Plots of hazard function of the BA distribution for selected values of the parameters α, θ, a and b

2.3 Series Expansion of the PDF and CDF of BA distribution

Using the series expansion

$$(1-z)^{a-1} = \sum_{i=1}^{\infty} \binom{a-1}{i} (-1)^i z^i, \tag{15}$$

the PDF of the BA distribution can be expanded as follows

$$f_{BA}(x) = \frac{1}{B(a,b)} \sum_{i=0}^{\infty} \binom{a-1}{i} (-1)^i \left[\bar{F}_A(x)\right]^{b-1} f_A(x) \tag{16}$$

and

$$f_{BA}(x) = \frac{1}{B(a,b)} \sum_{i=0}^{\infty} \binom{b-1}{i} \frac{(-1)^i}{a+i} f_{EA}(x; a+i, \theta) \tag{17}$$

where $f_A(x)$ is given in (2) and,

$$\bar{F}_A(x) = 1 - F_A(x) = \left(1 + \frac{\theta x(\theta x + 2)}{\theta^2 + 2}\right) e^{-\theta x} \tag{18}$$

and

$$f_{EA}(x; a+i, \theta) = \frac{(a+i)(1+x^2)e^{-\theta x}}{\theta^2 + 2} \left[1 - \left(1 + \frac{\theta x(\theta x + 2)}{\theta^2 + 2}\right) e^{-\theta x}\right]^{a+i-1} \tag{19}$$

The corresponding expansions for the CDF of the BA distribution are

$$F_{BA}(x)=1-\frac{1}{B(a,b)}\sum_{i=0}^{\infty}\binom{a-1}{i}\frac{(-1)^i}{b+i}\left(1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right)^{b+i}e^{-(b+i)\theta x} \quad (20)$$

and

$$F_{BA}(x)=\frac{1}{B(a,b)}\sum_{i=0}^{\infty}\binom{b-1}{i}\frac{(-1)^i}{a+i}F_{EA}(x;a+i,\theta) \quad (21)$$

where

$$F_{EA}(x;a+i,\theta)=\left[1-\left(1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right)e^{-\theta x}\right]^{a+i} \quad (22)$$

3. Properties of the BA distribution

3.1 Moment of the BA distribution and its related measures

If $a>0$, $b>0$ are real non-integers and X follows the BA distribution, then

$$E(X^r)=\int_0^{\infty}x^r f_{BA}(x;a,b,\theta)dx \quad (23)$$

Using (6) in (23) leads to

$$\mu_r'=E(X^r)=\frac{\theta^3}{B(a,b)(\theta^2+2)}\sum_{i=0}^{\infty}\binom{a-1}{i}(-1)^i\int_0^{\infty}x^r(1+x^2)\left(1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right)^{b+i-1}e^{-(b+i)\theta x}dx$$

$$=\frac{\theta^3}{B(a,b)(\theta^2+2)}\sum_{i=0}^{\infty}\sum_{j=0}^{b+i-1}\sum_{k=0}^{j}\binom{a-1}{i}\binom{b+i-1}{j}\binom{j}{k}(-1)^i \times\frac{2^k\theta^{2j-k}}{(\theta^2+2)^j}\left[\int_0^{\infty}\left(x^{r+2j-k}+x^{r+2j-k+2}\right)e^{-(b+i)\theta x}dx\right]$$

$$=\frac{\theta^3}{B(a,b)(\theta^2+2)}\sum_{i=0}^{\infty}\sum_{j=0}^{b+i-1}\sum_{k=0}^{j}\binom{a-1}{i}\binom{b+i-1}{j}\binom{j}{k}(-1)^i \times\frac{2^k\theta^{2j-k}}{(\theta^2+2)^j}\left[\frac{\Gamma(r+2j-k+1)}{[(b+i)\theta]^{r+2j-k+1}}+\frac{\Gamma(r+2j-k+3)}{[(b+i)\theta]^{r+2j-k+3}}\right]$$

$$\mu_r'=\frac{\theta^{-r}}{B(a,b)(\theta^2+2)}\sum_{i=0}^{\infty}\sum_{j=0}^{b+i-1}\sum_{k=0}^{j}\binom{a-1}{i}\binom{b+i-1}{j}\binom{j}{k}(-1)^i \times\frac{2^k\Gamma(r+2j-k+1)}{(\theta^2+2)^j(b+i)^{r+2j-k+1}}\left[\theta^2+\frac{(r+2j-k+2)(r+2j-k+1)}{(b+i)^2}\right] \quad (24)$$

Putting $r=1,2,3,\text{and}\,4$ into (24), we obtain the first four crude moments of the BA distribution as follows

$$\mu_1' = \frac{\theta^{-1}}{B(a,b)(\theta^2+2)} \sum_{i=0}^{\infty} \sum_{j=0}^{b+i-1} \sum_{k=0}^{j} \binom{a-1}{i} \binom{b+i-1}{j} \binom{j}{k} (-1)^i \times \frac{2^k \Gamma(2j-k+2)}{(\theta^2+2)^j (b+i)^{2j-k+2}} \left[\theta^2 + \frac{(2j-k+3)(2j-k+2)}{(b+i)^2} \right] \tag{25}$$

$$\mu_2' = \frac{\theta^{-2}}{B(a,b)(\theta^2+2)} \sum_{i=0}^{\infty} \sum_{j=0}^{b+i-1} \sum_{k=0}^{j} \binom{a-1}{i} \binom{b+i-1}{j} \binom{j}{k} (-1)^i \times \frac{2^k \Gamma(2j-k+3)}{(\theta^2+2)^j (b+i)^{2j-k+4}} \left[\theta^2 + \frac{(2j-k+4)(2j-k+3)}{(b+i)^2} \right] \tag{26}$$

$$\mu_3' = \frac{\theta^{-3}}{B(a,b)(\theta^2+2)} \sum_{i=0}^{\infty} \sum_{j=0}^{b+i-1} \sum_{k=0}^{j} \binom{a-1}{i} \binom{b+i-1}{j} \binom{j}{k} (-1)^i \times \frac{2^k \Gamma(2j-k+4)}{(\theta^2+2)^j (b+i)^{2j-k+4}} \left[\theta^2 + \frac{(2j-k+5)(2j-k+4)}{(b+i)^2} \right] \tag{27}$$

$$\mu_4' = \frac{\theta^{-4}}{B(a,b)(\theta^2+2)} \sum_{i=0}^{\infty} \sum_{j=0}^{b+i-1} \sum_{k=0}^{j} \binom{a-1}{i} \binom{b+i-1}{j} \binom{j}{k} (-1)^i \times \frac{2^k \Gamma(2j-k+5)}{(\theta^2+2)^j (b+i)^{2j-k+5}} \left[\theta^2 + \frac{(2j-k+6)(2j-k+5)}{(b+i)^2} \right] \tag{28}$$

The coefficients of variation (CV), skewness (α_3) and kurtosis (α_4) of the BA distribution could be obtained by evaluating the following:

$$CV = \sqrt{\frac{\mu_2'}{(\mu_2')^2} - 1} \tag{29}$$

$$\alpha_3 = \frac{\mu_3' - 3\mu_3'\mu_1' + 2(\mu_1')^3}{\left[\mu_2' - (\mu_1')^2\right]^{\frac{3}{2}}} \tag{30}$$

$$\alpha_4 = \frac{\mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4}{\left[\mu_2' - (\mu_1')^2\right]^2}. \tag{31}$$

3.2 Moment generating function of the BA distribution

Suppose $a > 0$, $b > 0$ are real non-integers and X follows the BA distribution, then moment generating function (mgf) of X is given by

$$M_X(t)=\int_0^\infty e^{tx} f_{BA}(x;a,b,\theta)dx \tag{32}$$

By inserting (6) into (32) and simplifying as in (24), the following is obtained

$$\begin{aligned}M_X(t)&=\frac{\theta^3}{B(a,b)(\theta^2+2)}\sum_{i=0}^{\infty}\sum_{j=0}^{b+i-1}\sum_{k=0}^{j}\binom{a-1}{i}\binom{b+i-1}{j}\binom{j}{k}(-1)^i\\&\times\frac{2^k\theta^{2j-k}}{(\theta^2+2)^j}\left[\int_0^\infty\left(x^{2j-k}+x^{2j-k+2}\right)e^{-((b+i)\theta-t)x}dx\right]\\&=\frac{\theta^3}{B(a,b)(\theta^2+2)}\sum_{i=0}^{\infty}\sum_{j=0}^{b+i-1}\sum_{k=0}^{j}\binom{a-1}{i}\binom{b+i-1}{j}\binom{j}{k}(-1)^i\\&\times\frac{2^k\theta^{2j-k}}{(\theta^2+2)^j}\left[\frac{\Gamma(2j-k+1)}{[(b+i)\theta-t]^{(2j-k+1)}}+\frac{\Gamma(2j-k+3)}{[(b+i)\theta-t]^{(2j-k+3)}}\right]\end{aligned}$$

$$\begin{aligned}M_X(t)&=\frac{1}{B(a,b)(\theta^2+2)}\sum_{i=0}^{\infty}\sum_{j=0}^{b+i-1}\sum_{k=0}^{j}\binom{a-1}{i}\binom{b+i-1}{j}\binom{j}{k}(-1)^i\\&\times\frac{2^k\Gamma(2j-k+1)}{(\theta^2+2)^j(b+i-t/\theta)^{(2j-k+1)}}\left[\theta^2+\frac{(2j-k+2)(2j-k+1)}{(b+i-t/\theta)^2}\right]\end{aligned} \tag{33}$$

3.3 Conditional moments of the BA distribution

Conditional moments provide platform for obtaining mean residual life function, mean deviations, Bonferroni and Lorenz curves, respectively. Let $X\sim BA(\theta,a,b)$, then nth conditional moment of X is given by

$$\begin{aligned}E_{BA}\left(X^n\middle|X>x\right)&=\frac{1}{1-F_{BA}(x)}\int_x^\infty y^n f_{BA}(y)dy\\&=\frac{\theta^3}{B(a,b)(\theta^2+2)}\sum_{i=0}^{\infty}\binom{a-1}{i}(-1)^i\int_0^\infty y^n(1+y^2)\left(1+\frac{\theta y(\theta y+2)}{\theta^2+2}\right)^{b+i-1}e^{-(b+i)\theta y}dy\\&=\sum_{i=0}^{\infty}\sum_{j=0}^{b+i-1}\sum_{k=0}^{j}(-1)^i\binom{a-1}{i}\binom{b+i-1}{j}\binom{j}{k}\frac{2^k\theta^{2j-k}}{B(a,b)(\theta^2+2)^{j+1}}\int_x^\infty\left(y^{2j-k+n}+y^{2(j+1)-k+n}\right)e^{-(b+i)\theta y}dy\end{aligned}$$

By letting $u=(b+i)\theta y$, $y=u/(b+i)\theta$ and $dy=du/(b+i)\theta$ yields

$$E_{BA}\left(X^n\middle|X>x\right)=\frac{1}{1-F_{BA}(x)}\left\{\begin{aligned}&\sum_{i=0}^{\infty}\sum_{j=0}^{b+i-1}\sum_{k=0}^{j}(-1)^i\binom{a-1}{i}\binom{b+i-1}{j}\binom{j}{k}\frac{2^k\theta^{2j-k}}{B(a,b)(\theta^2+2)^{j+1}}\\&\times\int_{\theta(b+i)x}^{\infty}\left[(u/(b+i)\theta)^{2j-k+n}+(u/(b+i)\theta)^{2(j+1)-k+n}\right]e^{-u}\,du/(b+i)\theta\end{aligned}\right\}$$

$$=\frac{1}{1-F_{BA}(x)}\sum_{i=0}^{\infty}\sum_{j=0}^{b+i-1}\sum_{k=0}^{j}\binom{a-1}{i}\binom{b+i-1}{j}\binom{j}{k}\frac{(-1)^i 2^k \theta^{-n}}{B(a,b)(\theta^2+2)^{j+1}(b+i)^{2j-k+n+3}} \times\left[(\theta(b+i))^2\Gamma(2j-k+n+1,\,\theta(b+i)x)+\Gamma(2j-k+n+3,\,\theta(b+i)x)\right] \quad (34)$$

where $1-F_{BA}(x)$ can be deduced from (5).

3.4 Mean residual life function of the BA distribution

The mean residual life function is given by

$$E_{BA}(X-x|X>x)=\frac{1}{1-F_{BA}(x)}\int_x^{\infty} y\, f_{BA}(y)dy-x$$
$$=\frac{1}{1-F_{BA}(x)}\left\{\sum_{i=0}^{\infty}\sum_{j=0}^{b+i-1}\sum_{k=0}^{j} w_{ijk}\left[(\theta(b+i))^2\Gamma(2j-k+2,\,\theta(b+i)x)+\Gamma(2j-k+4,\,\theta(b+i)x)\right]\right\}-x \quad (35)$$

3.5 Mean deviations

The mean deviation about the mean and the mean deviation about the median are defined by

$$\delta_1(x)=\int_0^{\infty}|x-\mu|\, f_{BA}(x)dx \text{ and } \delta_2(x)=\int_0^{\infty}|x-M|\, f_{BA}(x)dx,$$

respectively, where $\mu=E(X)$ and $M=Median(X)$ denote the mean and median of X. For computations, the following formulae are adopted:

$$\delta_1(x)=2\mu F_{BA}(\mu)-2\mu\int_{\mu}^{\infty} x\, f_{BA}(x)dx \text{ and } \delta_2(x)=2\int_{M}^{\infty} x\, f_{BA}(x)dx-\mu.$$

Thus,

$$\delta_1(x)=2\mu F_{BA}(\mu)-2\mu\sum_{i=0}^{\infty}\sum_{j=0}^{b+i-1}\sum_{k=0}^{j}\binom{a-1}{i}\binom{b+i-1}{j}\binom{j}{k}\frac{(-1)^i 2^k \theta^{-1}}{B(a,b)(\theta^2+2)^{j+1}(b+i)^{2j-k+4}} \times\left[(\theta(b+i))^2\Gamma(2j-k+2,\,\theta(b+i)\mu)+\Gamma(2j-k+4,\,\theta(b+i)\mu)\right] \quad (36)$$

$$\delta_2(x)=\left[\begin{array}{l}2\sum_{i=0}^{\infty}\sum_{j=0}^{b+i-1}\sum_{k=0}^{j}\binom{a-1}{i}\binom{b+i-1}{j}\binom{j}{k}\frac{(-1)^i 2^k \theta^{-1}}{B(a,b)(\theta^2+2)^{j+1}(b+i)^{2j-k+4}}\\ \times\left[(\theta(b+i))^2\Gamma(2j-k+2,\,\theta(b+i)M)+\Gamma(2j-k+4,\,\theta(b+i)M)\right]\end{array}\right]-\mu, \quad (37)$$

where $F_{BA}(\mu)$ and $F_{BA}(M)$ are easily calculated from (5) by replacing x with μ and M.

3.6 Bonferroni and Lorenz curves

According to Bonferroni (1930), the Bonferroni and Lorenz curves are defined as

$$B(p)=\frac{1}{p\mu}\int_0^q x\, f_{BA}(x)dx=\frac{1}{p\mu}\left(\mu-\int_q^{\infty} x\, f_{BA}(x)dx\right)$$

and

$$L(p)=\frac{1}{\mu}\int_0^q x\, f_{BA}(x)dx=\frac{1}{\mu}\left(\mu-\int_q^{\infty} x\, f_{BA}(x)dx\right)$$

Thus, the Bonferroni and Lorenz curves for the BA distribution are:

$$B(p)=\frac{1}{p\mu}\left[\mu-\left(\begin{array}{l}\sum_{i=0}^{\infty}\sum_{j=0}^{b+i-1}\sum_{k=0}^{j}\binom{a-1}{i}\binom{b+i-1}{j}\binom{j}{k}\frac{(-1)^i 2^k\theta^{-1}}{B(a,b)(\theta^2+2)^{j+1}(b+i)^{2j-k+4}}\\ \times\left((\theta(b+i))^2\Gamma(2j-k+2,\,\theta(b+i)q)+\Gamma(2j-k+4,\,\theta(b+i)q)\right)\end{array}\right)\right]. \quad (38)$$

$$L(p)=\frac{1}{\mu}\left[\mu-\left(\begin{array}{l}\sum_{i=0}^{\infty}\sum_{j=0}^{b+i-1}\sum_{k=0}^{j}\binom{a-1}{i}\binom{b+i-1}{j}\binom{j}{k}\frac{(-1)^i 2^k\theta^{-1}}{B(a,b)(\theta^2+2)^{j+1}(b+i)^{2j-k+4}}\\ \times\left[(\theta(b+i))^2\Gamma(2j-k+2,\,\theta(b+i)q)+\Gamma(2j-k+4,\,\theta(b+i)q)\right]\end{array}\right)\right]. \quad (39)$$

3.7 Entropies of the BA distribution

In this section, we derive the Rényi and Tsallis entropies for the BA distribution. Suppose that $\upsilon>0$ and $\upsilon\neq 1$, then the Rényi (Rényi, 1961), entropy of $X\sim BA(\theta,a,b)$ can be expressed as

$$I_R(\upsilon)=(1-\upsilon)^{-1}\log\left(\int_0^{\infty}(f_{BA}(x))^{\upsilon}dx\right)$$

$$=(1-\upsilon)^{-1}\log\left[\int_0^{\infty}\left(\frac{\theta^3(1+x^2)e^{-\theta bx}}{B(a,b)(\theta^2+2)}\left(1-\left(1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right)e^{-\theta x}\right)^{a-1}\left(1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right)^{b-1}\right)^{\upsilon}dx\right]$$

$$=(1-\upsilon)^{-1}\log\left[\begin{array}{l}\left(\frac{\theta^3}{B(a,b)(\theta^2+2)}\right)^{\upsilon}\int_0^{\infty}(1+x^2)^{\upsilon}\left(1-\left(1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right)e^{-\theta x}\right)^{\upsilon(a-1)}\\ \times\left(1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right)^{\upsilon(b-1)}e^{-\theta b\upsilon x}dx\end{array}\right] \quad (40)$$

Expanding the binomials in (40) and simplifying the integral (40) results to

$$I_R(\upsilon)=(1-\upsilon)^{-1}\log\left(\begin{array}{l}\sum_{i=0}^{\infty}\sum_{j=0}^{\upsilon(b-1)+i}\sum_{k=0}^{j}\sum_{l=0}^{\upsilon}\binom{\upsilon(a-1)}{i}\binom{\upsilon(b-1)+i}{j}\binom{j}{k}\binom{\upsilon}{l}\\ \times\frac{2^k\theta^{3\upsilon-2l}\Gamma(2j-k+2l+1)}{B(a,b)^{\upsilon}(\theta^2+2)^{\upsilon+j}[\upsilon(b+a-1)]^{2j-k+2l+1}}\end{array}\right) \quad (41)$$

Also, the Tsallis entropy of the BA distribution is given by

$$I_S(\upsilon)=(1-\upsilon)^{-1}\left(1-\int_0^{\infty}(f_{BA}(x))^{\upsilon}dx\right)$$

$$=(1-\upsilon)^{-1}\left[1-\left(\begin{array}{l}\sum_{i=0}^{\infty}\sum_{j=0}^{\upsilon(b-1)+i}\sum_{k=0}^{j}\sum_{l=0}^{\upsilon}\binom{\upsilon(a-1)}{i}\binom{\upsilon(b-1)+i}{j}\binom{j}{k}\binom{\upsilon}{l}\\ \times\frac{2^k\theta^{3\upsilon-2l}\Gamma(2j-k+2l+1)}{B(a,b)^{\upsilon}(\theta^2+2)^{\upsilon+j}[\upsilon(b+a-1)]^{2j-k+2l+1}}\end{array}\right)\right]. \quad (42)$$

3.8 Distribution of order statistics

Suppose $X_1,X_2,...,X_n$ constitutes a BA distribution random sample of size n. Let the corresponding order statistics be $X_{1:n}\leq X_{2:m}\leq ...\leq X_{n:n}$. Then the pdf of the mth order statistic $X_{m:n}$ can be expressed as

$$f_{X_{m:n}}(x)=\frac{n!\,f_{BA}(x)}{(m-1)!(n-m)!}\sum_{s=0}^{m-1}\binom{m-1}{s}(-1)^s\left(1-F_{BA}(x)\right)^{n-m} \tag{43}$$

Substituting (6) and (20) into (43) yields Rényi

$$\begin{aligned}f_{X_{m:n}}(x)=&\frac{n!\theta^3\left(1+x^2\right)e^{-\theta bx}}{B(a,b)\left(\theta^2+2\right)(m-1)!(n-m)!}\left(1-\left(1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right)e^{-\theta x}\right)^{a-1}\left(1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right)^{b-1}\\&\times\sum_{s=0}^{m-1}\binom{m-1}{s}(-1)^s\left(\frac{1}{B(a,b)}\sum_{i=0}^{\infty}\binom{a-1}{i}\frac{(-1)^i}{b+i}\left(1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right)^{b+i}e^{-(b+i)\theta x}\right)^{n-m+s}\end{aligned} \tag{44}$$

The first and largest order statistics of the BA distribution becomes

$$\begin{aligned}f_{X_{1:n}}(x)=&\frac{n\theta^3\left(1+x^2\right)e^{-\theta bx}}{B(a,b)\left(\theta^2+2\right)}\left(1-\left(1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right)e^{-\theta x}\right)^{a-1}\left(1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right)^{b-1}\\&\times\left(\frac{1}{B(a,b)}\sum_{i=0}^{\infty}\binom{a-1}{i}\frac{(-1)^i}{b+i}\left(1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right)^{b+i}e^{-(b+i)\theta x}\right)^{n-1}\end{aligned} \tag{45}$$

and

$$\begin{aligned}f_{X_{n:n}}(x)=&\frac{n\theta^3\left(1+x^2\right)e^{-\theta bx}}{B(a,b)\left(\theta^2+2\right)}\left(1-\left(1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right)e^{-\theta x}\right)^{a-1}\left(1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right)^{b-1}\\&\times\left(1-\frac{1}{B(a,b)}\sum_{i=0}^{\infty}\binom{a-1}{i}\frac{(-1)^i}{b+i}\left(1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right)^{b+i}e^{-(b+i)\theta x}\right)^{n-1}\end{aligned} \tag{46}$$

The corresponding cdf of the mth order statistic is

$$\begin{aligned}F_{X_{m:n}}(x)&=\sum_{i=s}^{n}\sum_{d=0}^{n-i}\binom{n}{i}\binom{n-i}{d}\left[F_{BE}(x)\right]^{i+d}\\&=\sum_{i=s}^{n}\sum_{d=0}^{n-i}\binom{n}{i}\binom{n-i}{d}\left(\frac{1}{B(a,b)}\sum_{i=0}^{\infty}\binom{a-1}{i}\frac{(-1)^i}{b+i}\left(1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right)^{b+i}e^{-(b+i)\theta x}\right)^{i+d}\end{aligned} \tag{47}$$

3.9 Maximum likelihood estimation of parameters of the BA distribution

The likelihood function of a random sample $X_1,X_2,...,X_n$ drawn from a BA distribution may be written as

$$L=\left(\frac{\theta^3}{B(a,b)\left(\theta^2+2\right)}\right)^n e^{-\theta b\sum_{i=1}^{n}x_i}\prod_{i=1}^{n}\left(1+x_i^2\right)\left(1-\left(1+\frac{\theta x_i(\theta x_i+2)}{\theta^2+2}\right)e^{-\theta x_i}\right)^{a-1}\left(1+\frac{\theta x_i(\theta x_i+2)}{\theta^2+2}\right)^{b-1} \tag{48}$$

The log-likelihood function of (48) becomes

$$\ln L = -n\ln(B(a,b)) + (a-1)\sum_{i=1}^{n}\ln\left(1-\left(1+\frac{\theta x_i(\theta x_i+2)}{\theta^2+2}\right)e^{-\theta x_i}\right) + (b-1)\sum_{i=1}^{n}\ln\left(1+\frac{\theta x_i(\theta x_i+2)}{\theta^2+2}\right)$$
$$+n\ln\left(\frac{\theta^3}{\theta^2+2}\right) - \theta b\sum_{i=1}^{n}x_i + \sum_{i=1}^{n}\ln(1+x_i^2) \quad (49)$$

From (49), the following log-likelihood equations are obtained:

$$\frac{\partial \ln L}{\partial a} = -n\psi(a) + n\psi(a+b) + \sum_{i=1}^{n}\ln\left(1-\left(1+\frac{\theta x_i(\theta x_i+2)}{\theta^3+2}\right)e^{-\theta x_i}\right) = 0 \quad (50)$$

$$\frac{\partial \ln L}{\partial b} = -n\psi(b) + n\psi(a+b) + \sum_{i=1}^{n}\ln\left(1+\frac{\theta x_i(\theta x_i+2)}{\theta^3+2}\right) - \theta\sum_{i=1}^{n}x_i = 0 \quad (51)$$

$$\frac{\partial \ln L}{\partial \theta} = (a-1)\sum_{i=1}^{n}\frac{\left(\theta^4+6\theta^2+\theta^4 x_i^2+2\theta^3 x_i+2\theta^2 x_i^2\right)x_i e^{-\theta x_i}}{(\theta^2+2)\left(\theta^2+2-\left(\theta^2+2+\theta x_i(\theta x_i+2)\right)e^{-\theta x_i}\right)} + \frac{n(\theta^2+6)}{\theta(\theta^2+2)}$$
$$-2(b-1)\sum_{i=1}^{n}\frac{(\theta^2-2\theta x_i-2)}{(\theta^2+2)(\theta^2+2+\theta x_i(\theta x_i+2))} - b\sum_{i=1}^{n}x_i = 0 \quad (52)$$

The solution of (50)-(52) gives the maximum likelihood estimators of the parameters of the BA distribution. However, due to the complexity of (50)-(51), the R package was used to resolve the equations.

3.10 Asymptotic confidence intervals of the parameters of BA distribution

Let $\hat{\Theta} = \left(\hat{a},\hat{b},\hat{\theta}\right)^T$ be the MLE of $\Theta = \left(a,b,\theta\right)^T$ for the BA distribution. To construct the confidence intervals, we need the Fisher information matrix, denoted by $\boldsymbol{I}(\Theta)$. Thus,

$$\boldsymbol{I}(\Theta) = \begin{pmatrix} \boldsymbol{I}_{aa} & \boldsymbol{I}_{ab} & \boldsymbol{I}_{a\theta} \\ \boldsymbol{I}_{ab} & \boldsymbol{I}_{bb} & \boldsymbol{I}_{b\theta} \\ \boldsymbol{I}_{a\theta} & \boldsymbol{I}_{b\theta} & \boldsymbol{I}_{\theta\theta} \end{pmatrix} \quad (53)$$

The entries of the Hessian matrix (53) are given as follows

$$I_{aa} = \frac{\partial^2 \ln L}{\partial a^2} = n\psi'(a+b) - n\psi'(a) \quad (54)$$

$$I_{bb} = \frac{\partial^2 \ln L}{\partial b^2} = n\psi'(a+b) - n\psi'(b) \quad (55)$$

$$I_{ab} = \frac{\partial^2 \ln L}{\partial a \partial b} = n\psi'(a+b) \quad (56)$$

$$I_{a\theta} = \frac{\partial^2 \ln L}{\partial \theta^2} = \sum_{i=1}^{n}\frac{\left(\theta^4+6\theta^2+\theta^4 x_i^2+2\theta^3 x_i+2\theta^2 x_i^2\right)x_i e^{-\theta x_i}}{(\theta^2+2)\left(\theta^2+2-\left(\theta^2+2+\theta x_i(\theta x_i+2)\right)e^{-\theta x_i}\right)} \quad (57)$$

$$I_{b\theta} = \frac{\partial^2 \ln L}{\partial b \partial \theta} = -2\sum_{i=1}^{n}\frac{x_i(\theta^2-2\theta x_i-2)}{(\theta^2+2)(\theta^2+2+\theta x_i(\theta x_i+2))} - \sum_{i=1}^{n}x_i \quad (58)$$

$$I_{\theta\theta}=\frac{\partial^2 \ln L}{\partial\theta^2}=(a-1)\sum_{i=1}^{n}\frac{v_1(x_i,\theta)u_1^{'}(x_i,\theta)-u_1(x_i,\theta)v_1^{'}(x_i,\theta)}{v_1^2(x_i,\theta)}-\frac{n\left(\theta^4+16\theta^2+12\right)}{\theta^2\left(\theta^2+2\right)^2}$$
$$-2(b-1)\sum_{i=1}^{n}\frac{v_2(x_i,\theta)u_2^{'}(x_i,\theta)-u_2(x_i,\theta)v_2^{'}(x_i,\theta)}{v_2^2(x_i,\theta)} \quad (59)$$

where

$$u_1(x_i,\theta)=\left(1+\frac{2\left(\theta^2-2\theta x_i-2\right)}{\left(\theta^2+2\right)^2}+\frac{\theta x_i(\theta x_i+2)}{\theta^2+2}\right)x_i e^{-\theta x_i} \quad (60)$$

$$v_1(x_i,\theta)=1-\left(1+\frac{\theta x_i(\theta x_i+2)}{\theta^2+2}\right)e^{-\theta x_i} \quad (61)$$

$$u_1^{'}(x_i,\theta)=-2\left(\frac{2\theta^3+6\theta^2x_i^2+8\theta x_i-4x_i^2+\theta^4x_i-2\theta^2x_i^2-4}{\left(\theta^2+2\right)^3}\right)x_i e^{-\theta x_i}$$
$$-\left(1+\frac{\theta^4x_i^2+2\theta^3x_i+2\theta^2x_i^2+2\theta^2x_i-4}{\left(\theta^2+2\right)^2}\right)x_i^2 e^{-\theta x_i} \quad (62)$$

$$v_1^{\prime}(x_i,\theta)=\left(\frac{\theta^4+6\theta^2+\theta^4x_i^2+2\theta^3x_i+2\theta^2x_i^2}{\left(\theta^2+2\right)^2}\right)x_i e^{-\theta x_i} \quad (63)$$

$$u_2(x_i,\theta)=\frac{2x_i\left(\theta^2-2\theta x_i-2\right)}{\left(\theta^2+2\right)^2} \quad (64)$$

$$v_2(x_i,\theta)=1+\frac{\theta x_i(\theta x_i+2)}{\theta^2+2} \quad (65)$$

$$u_2^{\prime}(x_i,\theta)=\frac{-4\theta^3x_i-12\theta^2x_i^2+24\theta x_i-8x_i^2}{\left(\theta^2+2\right)^3} \quad (66)$$

$$v_2^{\prime}(x_i,\theta)=\frac{4\theta x_i^2+6\theta^2x_i+4x_i+}{\left(\theta^2+2\right)^2} \quad (67)$$

Under certain regularity conditions (see, for example, Lehmann and Casella (1998), the asymptotic distribution of $\sqrt{n}\left(\hat{\boldsymbol{\Theta}}_{-}\boldsymbol{\Theta}\right)$ is $N_4\left(\mathbf{0},\ \boldsymbol{I}^{-1}(\boldsymbol{\Theta})\right)$. The asymptotic multivariate normal distribution with mean vector $(0,0,0,\ 0)^T$ and covariance matrix $\boldsymbol{I}^{-1}(\boldsymbol{\Theta})$ can used to construct approximate confidence intervals the model parameters. Consequently, the approximate $100(1-\tau)\%$ two-sided confidence intervals for α, θ, a and b are given, respectively, by

$$\hat{a}\pm Z_{\tau/2}\sqrt{I_{aa}^{-1}\left(\hat{\Theta}\right)} \quad (68)$$

$$\hat{b}\pm Z_{\tau/2}\sqrt{I_{bb}^{-1}\left(\hat{\Theta}\right)} \quad (69)$$

$$\hat{\theta}\pm Z_{\tau/2}\sqrt{I_{\theta\theta}^{-1}\left(\hat{\Theta}\right)}, \quad (70)$$

where $I_{\theta\theta}^{-1}\left(\hat{\Theta}\right)$, $I_{aa}^{-1}\left(\hat{\Theta}\right)$ and $I_{bb}^{-1}\left(\hat{\Theta}\right)$ are the diagonal elements of the matrix $\boldsymbol{I}_n^{-1}\left(\hat{\boldsymbol{\Theta}}\right)$ and $Z_{\tau/2}$ is the upper $(\tau/2)$th percentile of a standard normal distribution.

4. Application

The performance of the BA distribution vis-à-vis other related distributions considered in this study in fitting two real data sets is demonstrated in this section. The estimation of the parameters of each of the distributions is made using the maximum likelihood method of estimation. The standard error of each of the parameters estimated is enclosed in brackets. The performance measures obtained include Log-likelihood (LL), Akaike's information criterion (AIC), Kolmogorov-smirnov (K-S) statistic, and the corresponding probability value (p-value). The comparison of the proposed distribution is conducted with some well-known lifetime distributions such as the Ishita distribution (ID), Exponentiated Ishita distribution (EID), Akash distribution (AD), Exponentiated Akash distribution (EAD), Exponential distribution (ED) and Exponentiated Exponential distribution (EED), respectively.

Dataset

The data represents the sum of skin folds in 202 athletes collected at the Australian Institute of sports and used in a book by Weisberg (2005). The data is given below:

28.0, 98.0, 89.0, 68.9, 69.9, 109.0, 52.3, 52.8, 46.7, 82.7, 42.3, 109.1, 96.8, 98.3, 103.6, 110.2, 98.1, 57.0, 43.1, 71.1, 29.7, 96.3, 102.8, 80.3, 122.1, 71.3, 200.8, 80.6, 65.3, 78.0, 65.9, 38.9, 56.5, 104.6, 74.9, 90.4, 54.6, 131.9, 68.3, 52.0, 40.8, 34.3, 44.8, 105.7, 126.4, 83.0, 106.9, 88.2, 33.8, 47.6, 42.7, 41.5, 34.6, 30.9, 100.7, 80.3, 91.0, 156.6, 95.4, 43.5, 61.9, 35.2, 50.9, 31.8, 44.0, 56.8, 75.2, 76.2, 101.1, 47.5, 46.2, 38.2, 49.2, 49.6, 34.5, 37.5, 75.9, 87.2, 52.6, 126.4, 55.6, 73.9, 43.5, 61.8, 88.9, 31.0, 37.6, 52.8, 97.9, 111.1, 114.0, 62.9, 36.8, 56.8, 46.5, 48.3, 32.6, 31.7, 47.8, 75.1, 110.7, 70.0, 52.5, 67.0, 41.6, 34.8, 61.8, 31.5, 36.6, 76.0, 65.1, 74.7, 77.0, 62.6, 41.1, 58.9, 60.2, 43.0, 32.6, 48.0, 61.2, 171.1, 113.5, 148.9, 49.9, 59.4, 44.5, 48.1, 61.1, 31.0, 41.9, 75.6, 76.8, 99.8, 80.1, 57.9, 48.4, 41.8, 44.5, 43.8, 33.7, 30.9, 43.3, 117.8, 80.3, 156.6, 109.6, 50.0, 33.7, 54.0, 54.2, 30.3, 52.8, 49.5, 90.2, 109.5, 115.9, 98.5, 54.6, 50.9, 44.7, 41.8, 38.0, 43.2, 70.0, 97.2, 123.6, 181.7, 136.3, 42.3, 40.5, 64.9, 34.1, 55.7, 113.5, 75.7, 99.9, 91.2, 71.6, 103.6, 46.1, 51.2, 43.8, 30.5, 37.5, 96.9, 57.7, 125.9, 49.0, 143.5, 102.8, 46.3, 54.4, 58.3, 34.0, 112.5, 49.3, 67.2, 56.5, 47.6, 60.4, 34.9

The estimated parameters, their standard errors and confidence intervals for data set 1 are given in Table 1. Table 2 gives the Log-likelihood values, K-S statistic and their p-values, AIC and BIC values for data set.

Table 1: The MLEs of the parameters of the fitted distributions and their confidence intervals

Distribution	Parameter	MLE	Standard Error	Lower Bound	Upper Bound
BAD	θ	0.1459	0.0301	0.0869	0.2049
	a	7.5302	3.9159	-0.1450	15.2054
	b	0.0259	0.0780	-0.1270	0.1718
ID	θ	0.0435	0.0018	0.03997	0.0470
EID	α	0.6210	0.0417	0.53927	0.7027
	θ	0.0351	0.0021	0.03098	0.0392
AD	θ	0.0434	0.0018	0.03987	0.0469
EAD	α	2.1150	0.2815	1.56326	2.6667
	θ	0.0584	0.0034	0.05174	0.0651
LD	θ	0.0286	0.0014	0.02586	0.0313
ELD	α	3.7245	0.5294	2.68688	4.7621
	θ	0.0498	0.0030	0.04392	0.0557
ED	θ	0.0145	0.0010	0.01254	0.0165
EED	α	8.5952	1.3112	6.02525	11.1652
	θ	0.0407	0.0027	0.03541	0.0460

Table 2: The results of the log-lik, AIC, BIC, KS statistic and p-value of the fitted distributions

Disdtribution	Log-lik	KS Statistic	P-value	AIC	BIC
BAD	-952.1441	0.0684	0.2872	1910.2880	1920.213
AD	-976.1313	0.1373	0.0009	1954.2630	1957.5710
EID	-1024.0920	0.2183	0.0000	2052.1840	2058.8000
ID	-976.0187	0.1369	0.0009	1954.0370	1957.3460
EAD	-960.5972	0.2638	0.0000	1925.1940	1931.8110
LD	-1001.7430	0.2154	0.0000	2005.4860	2008.7950
ELD	-959.5830	0.0878	0.0836	1923.1660	1929.7830
ED	-1057.3530	0.3458	0.0000	2116.7070	2120.0150
EED	-958.0064	0.6986	0.0000	1920.0130	1926.6290

The results reported in Table 2 reveal that, on the average, the values of AIC, BIC and K-S are smaller for the BA distribution than the other distributions while the values of log-likelihood (LL) and p-values are higher for the BA than the other distributions. Hence, the BA distribution outperforms the other distributions with respect to data set used in this study.

5. Conclusion

This article proposed a new probability distribution called the Beta Akash (BA) distribution for lifetime analysis. We provide for the new distribution expressions for the distribution function, density function, moments, cumulants, skewness, kurtosis, moment generating function, characteristic function, mean deviations, entropies, Bonferroni and Lorenz curves, stress-strength reliability, distribution of order statistics and its moments. Estimation of the parameters by maximum likelihood method is discussed. An application to real lifetime data shows that the BA distribution is more flexible than its competitors and can be used quite effectively in analysing positively skewed and heavy tailed data.

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