Optimal and efficient eighteen-treatment semi-Latin squares in blocks of size three

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Semi-Latin squares with side six and block size three, constructed by superimposing each of the 9408 reduced Latin squares of order six on each of certain $(6 \times 6)/2$ semi-Latin squares, are here presented. The aim was to identify optimal and/or efficient semi-Latin square(s) of order six from the 9408 reduced Latin squares of order six. A Microsoft Excel program was used to facilitate the construction by superposition and the statistical evaluation of the corresponding semi-Latin squares of sides six in blocks of size three by computing their A, D, E and MV statistical efficiency measures A total of 65856 semi-Latin squares with side six and blocks of size three were constructed and evaluated. One of the semi-Latin squares was identified to be A-optimal, D-optimal, E-optimal and MV-optimal. Also, with respect to the efficiency measures, the same optimal semi-Latin square is the most efficient of the 65856 semi-Latin squares. This efficient semi-Latin square which has canonical efficiency factors, 0.5980 with multiplicity three, 0.6667 with multiplicity ten, 0.8464 with multiplicity three and 1.0 with multiplicity one, is a simple orthogonal multi-array (SOMA) of order six; specifically denoted by SOMA(3, 6). Also, this optimal and efficient semi-Latin square has the same A, D, E and MV statistical efficiency values with the two indecomposable SOMA(3, 6) developed by Soicher as well as the efficient semi-Latin square of order six by Bailey.

Keywords: canonical efficiency; contrasts; incidence matrix; reduced Latin squares; regular graph design; superposition

1. Introduction

An $(n \times n)/k$ semi-Latin square with n rows, n columns and k plots, according to Bailey and Chigbu (1997) and Soicher (2012a), has n^2 cells (called blocks) in a square array such that each block has k plots (letters) such that there are $t = nk$ treatments which are allocated to the plots in such a way that each treatment occurs once in each row and once in each column. We note that a semi-Latin square is considered suitable for comparing t treatments if the rows and columns of the $(n \times n)/k$ array are considered as nuisance factors (Bailey and Royle, 1997). When k Latin squares, each of order n, are given, their superposition is the semi-Latin square with parameters, n, k and t, formed by putting into each cell of the semi-Latin square, k mutually disjoint treatments out of the t treatments. Bailey (1990) constructed twelve-treatment semi-Latin squares in blocks of size two with the view to obtaining an efficient semi-Latin square for the parameters, $n = 6$ and $k = 2$. This was achieved by considering the twelve faces of a dodecahedron which forms an association scheme with three associate classes. The adjacency matrices of the inherent incomplete-block designs of the semi-Latin squares and their corresponding eigenvalues were used to estimate the efficiency factors of the semi-Latin squares. It was recommended in Bailey (1990) that the upper bound for the efficiency factor of a $(6 \times 6)/2$ semi-Latin square should be 0.5238, which is the harmonic mean efficiency factor of an optimal $(6 \times 6)/2$ Trojan square, if it

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were to exist.

Also, Bailey and Royle (1997) examined the twelve-treatment semi-Latin squares in blocks of size two by considering only those semi-Latin squares whose quotient block designs are regular-graph designs since it is widely believed that all optimal regular-graph designs are optimal overall. The designs were constructed using orthogonal one-factorizations of the regular graph design. This procedure is based on the fact that the one-factorizations of the rows and columns of a semi-Latin square of a regular graph have the property that they are orthogonal. In this regard, the identification of one-factor leads to the location of its orthogonal mate.

Consider, for instance, a regular graph design, Ω, of degree six on twelve vertices as the treatment-concurrence graph of a semi-Latin square, Λ, where the thirty-six treatment entries of Λ are determined solely by the thirty-six edges of Ω. The graph, Ω, is regular if all vertices are of the same degree (Kreher *et al.*, 1996). The concepts of one-factor and one-factorization were instrumental in realizing the treatment-concurrence graph by Bailey and Royle (1997). Also, Meszka and Tyniec (2019) defined one-factor of a graph, Ω, as a regular spanning subgraph of degree one while a one-factorization of Ω is the partition of the edges of Ω into one-factors. Precisely, each row and column of Λ is a one-factor of Ω while the collections of rows of Λ is a one-factorization and so also is the collection of columns of Λ. In order to have a one-factorization, Ω must have even number of vertices and must be regular. Bailey and Royle (1997) used two orthogonal one-factorizations of a regular graph design, Ω, in the search for optimal semi-Latin square of side six and block size two. Each one-factorization is the orthogonal mate of the other since they have one common edge. For further discussions on the orthogonality of two one-factorizations of a regular graph design, see Meszka and Tyniec (2019).

In the works of Bailey and Royle (1997), seven distinct $(6 \times 6)/2$ semi-Latin squares (see Figures 1 – 7) were in focus. These seven semi-Latin squares were evaluated using the *A*, *D*, *E* and *MV* criteria to ascertain their efficiencies. Any of the designs which maximize any of the efficiency criteria was considered to be optimal. Hence, the optimal $(6 \times 6)/2$ semi-Latin squares, with respect to each of the *A*, *D*, *E* and *MV* efficiency criteria, were further classified in such a way that the best three semi-Latin squares with the highest *A*-optimal values belong to one class, the best three with the highest *D*-optimal values belong to another class, and so on.

The first set of $(6 \times 6)/3$ semi-Latin squares were classified by Phillips and Wallis (1996) up to permutations of rows and columns, and renaming symbols; that is, strong isomorphism. There was, however, a classification mistake in Phillips and Wallis (1996) which led to the omission of one of the classes while repeating a particular class twice. This was pointed out in the application of the $(6 \times 6)/3$ semi-Latin squares to tournament problems by Preece and Phillips (2002). Also, Soicher (2012a) also presented another classification of efficient $(6 \times 6)/3$ semi-Latin squares, which he obtained independently in 1997 as simple orthogonal multi-arrays (SOMA). Soicher (2012a) presented four distinct efficient $(6 \times 6)/3$ semi-Latin squares with their *A*, *D*, *E* and *MV* statistical efficiency measures, and grouped the four $(6 \times 6)/3$ semi-Latin squares into two decomposable and two indecomposable SOMAs. One of the optimal $(6 \times 6)/3$ semi-Latin squares of Soicher (2012a) was made by superimposing a (6×6) standard Latin square presented as Figure 13 in Bailey (1992) on a $(6 \times 6)/2$ semi-Latin square, also presented as Figure 12 in Bailey (1992).

In this paper, we present another efficient $(6 \times 6)/3$ semi-Latin square which has the same values of the *A*, *D*, *E* and *MV* statistical efficiency measures of the two decomposable SOMAs presented by Soicher (2012a). The $(6 \times 6)/3$ semi-Latin squares given in this work were constructed by superimposing each of the reduced (6×6) Latin squares on each of the seven $(6 \times 6)/2$ semi-Latin squares of Bailey and Royle (1997).

2. Construction Procedure for $(6 \times 6)/3$ semi-Latin Squares

The seven distinct $(6 \times 6)/2$ semi-Latin squares studied by Bailey and Royle (1997) are pivotal in the construction of the $(6 \times 6)/3$ semi-Latin squares here and are identified in this study as $\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5, \Delta_6$ and Δ_7 (see Figures 1, 2, 3, 4, 5, 6 and 7, respectively).

6 15	8 17	10 7	12 9	14 11	16 13
8 10	13 15	12 14	17 7	16 6	9 11
11 13	10 12	15 17	14 16	7 9	6 8
9 14	11 16	13 6	15 8	17 10	7 12
7 16	9 6	11 8	13 10	15 12	17 14
12 17	14 7	16 9	6 11	8 13	10 15

Figure 1: The Howell Design, Δ_1

6 7	14 16	10 11	9 12	8 17	13 15
9 11	6 8	13 16	7 14	12 15	10 17
12 14	11 15	6 9	13 17	7 10	8 16
15 17	7 13	8 12	6 10	9 16	11 14
8 13	10 12	14 17	15 16	6 11	7 9
10 16	9 17	7 15	8 11	13 14	6 12

Figure 2: The $(6 \times 6)/2$ semi-Latin square, Δ_2

6 7	11 15	12 17	8 14	9 13	10 16
11 16	6 8	13 15	9 12	10 14	7 17
10 15	12 14	6 9	11 17	7 16	8 13
9 17	7 13	8 16	6 10	12 15	11 14
8 12	10 17	7 14	13 16	6 11	9 15
13 14	9 16	10 11	7 15	8 17	6 12

Figure 3: The $(6 \times 6)/2$ semi-Latin square, Δ_3

6 7	11 16	8 13	10 15	12 17	9 14

10 17	8 9	7 15	14 16	11 13	6 12
9 12	6 14	10 11	7 17	8 16	13 15
11 14	15 17	6 16	12 13	7 9	8 10
13 16	10 12	9 17	6 8	14 15	7 11
8 15	7 13	12 14	9 11	6 10	16 17

Figure 4: The $(6 \times 6)/2$ semi-Latin square,Δ_4

6 11	10 16	9 14	8 13	12 15	7 17
10 15	8 14	6 13	12 17	7 16	9 11
7 14	6 12	10 17	9 16	8 11	13 15
8 17	7 13	11 16	6 15	9 10	12 14
9 12	11 17	8 15	7 10	13 14	6 16
13 16	9 15	7 12	11 14	6 17	8 10

Figure 5: The $(6 \times 6)/2$ semi-Latin square, Δ_5

6 7	13 15	8 16	11 14	10 12	9 17
14 16	6 8	11 17	9 12	7 15	10 13
8 12	10 16	6 9	7 17	13 14	11 15
13 17	7 11	12 15	6 10	9 16	8 14
9 15	12 17	10 14	8 13	6 11	7 16
10 11	9 14	7 13	15 16	8 17	6 12

Figure 6: The $(6 \times 6)/2$ semi-Latin square, Δ_6

6 7	12 13	14 16	8 17	9 15	10 11
13 16	6 8	7 15	11 14	10 12	9 17
8 11	10 16	6 9	12 15	13 17	7 14
14 17	9 11	8 12	6 10	7 16	13 15
9 12	14 15	10 17	7 13	6 11	8 16
10 15	7 17	11 13	9 16	8 14	6 12

Figure 7: The $(6 \times 6)/2$ semi-Latin square,Δ_7

The semi-Latin square, Δ_1, called Howell design, was found by Hung and Mendelsohn (1974) and has been studied by Bailey (1997). Seah and Stinson (1987) developed Δ_4 and Δ_7 as rectangular association schemes of Δ_1. Based on these original works of Hung and Mendelsohn (1974) and Seah and Stinson (1987), Bailey and Royle (1997) constructed the other designs, Δ_2, Δ_3, Δ_5 and Δ_6, using orthogonal one- factorizations. Soicher (2013) constructed optimal and efficient $(6 \times 6)/k$ semi-Latin squares for k = 4, 5 and 6, since Soicher (2012a) had already addressed the case of efficient $(6 \times 6)/k$ semi-Latin square for k = 3. Also, Soicher (2013) provided exact algebraic approaches for comparing the efficiency measures of the semi-Latin squares. In this study, we construct all the $(6 \times 6)/3$ semi-Latin squares using all the reduced/standard Latin squares in combination with certain $(6 \times 6)/2$ semi-Latin squares in the literature via superimpositions, and determine the A, D, E and MV efficient ones.

We define a Latin square of order n as an $(n \times n)$ matrix, L, whose entries are taken from a set, S, of n symbols (treatments) and which has the property that each symbol from S occurs exactly once in each row and exactly once in each column of *L*. A Latin square, *L*, based on the natural ordering of numbers as symbols is called a reduced or a standard Latin square Denes and Keedwell (1991). There exists 9408 reduced (6×6) Latin square McKay (2019). A typical (6×6) reduced Latin square, denoted by Δ_0, is shown in Figure 8.

0	1	2	3	4	5
1	0	3	2	5	4
2	3	4	5	0	1
3	2	5	4	1	0
4	5	0	1	2	3
5	4	1	0	3	2

Figure 8: (6×6) Latin square, Δ_0

The combinatorial composition of reduced Latin squares of order six was given by McKay and Wanless (2005) as $2^6.3.7^2$ which gives the 9408 reduced Latin squares. Therefore, the total number of semi-Latin squares of order six in blocks of size three from the seven distinct $(6 \times 6)/2$ semi-Latin squares, is obtained as $2^6.3.7^3$, which gives 65856. Each of the 9408 reduced (6×6) Latin square was superimposed on each of the seven $(6 \times 6)/2$ semi-Latin squares of Bailey and Royle (1997) to obtain all possible 65856 $(6 \times 6)/3$ semi-Latin squares. For instance, superimposing Δ_0 on Δ_1 gives the $(6 \times 6)/3$ semi-Latin square, Δ_{01}, presented in Figure 9. A program that runs as a Microsoft Office version 16 Excel Macro developed by Chigbu, Mba and Ukaegbu (2021) was used to facilitate the construction of the $(6 \times 6)/3$ semi-Latin squares as well as the *A*, *D*, *E* and *MV* efficiency evaluations which took about half-hour of CPU time on Lion Grade, University of Nigeria main server.

0 6 15	1 8 17	2 10 7	3 12 9	4 14 11	5 16 13
1 8 10	0 13 15	3 12 14	2 17 7	5 16 6	4 9 11
2 11 13	3 10 12	4 15 17	5 14 16	0 7 9	1 6 8
3 9 14	2 11 16	5 13 6	4 15 8	1 17 10	0 7 12
4 7 16	5 9 6	0 11 8	1 13 10	2 15 12	3 17 14
5 12 17	4 14 7	1 16 9	0 6 11	3 8 13	2 10 15

Figure 9: The $(6 \times 6)/3$ semi-Latin squares, Δ_{01}

3. Evaluation for Efficient $(6 \times 6)/3$ semi-Latin squares

First of all, the incidence matrix of each of the 65856 $(6 \times 6)/3$ semi-Latin squares, labelled $N_1, N_2, \ldots, N_{65856}$, was obtained by $t \times n^2$ treatments-by-block matrix whose entry in the t^{th} row and b^{th} column is the number of times that each of *t* treatments occurs in each of the *b* blocks. Here, the treatments take the values, *t* = 0, 1, 2, . . ., 17 while the blocks take the values, *b* = 1, 2, …, 36. That is, for the entries of the incidence matrix, each block, where treatment, *t*, occurs is represented by 1 while where the treatment, *t*, does not occur is represented by 0, such that the incidence matrix is binary. The information matrices were denoted by $C_1, C_2, \ldots, C_{65856}$ and obtained by the formula,

$$C_p = R_p - N_p K_p^{-1} N_p^1, p = 1, 2, \ldots, 65856 \qquad (1)$$

where $R_p = rI$ is a diagonal matrix whose diagonal entries are the number of replications, r, of each treatment and I is an identity matrix of size n; N_p is the incidence matrix described above and $K_p = kI$ is also a diagonal matrix whose diagonal entries are block sizes, k.

The canonical efficiency factors of C_p are the non-zero eigenvalues of the information matrix, C_p. They are obtained by solving for the roots, λ, of the equation,

$$|C_p - \lambda I| = 0 \quad (2)$$

and obtaining all the non-zero values of λ/r. The efficiency measures, A, D and E are, respectively, the harmonic mean, geometric mean and minimum of the canonical efficiency factors. The *MV*-efficiency is not derived from the canonical efficiency factors but it is the minimum of the efficiency factors for simple contrasts (Bailey and Royle, 1997). Out of the 65856 $(6 \times 6)/3$ semi-Latin squares, one design has the highest values for each of the four efficiency criteria. This design is given in Figure 10 and it is the most efficient among the $(6 \times 6)/3$ semi-Latin squares, with the highest respective *A*, *D*, *E* and *MV* values of 0.6922, 0.6986, 0.5980 and 0.6586. This semi-Latin square has the same values of the statistical efficiency measures, *A*, *D*, *E* and *MV*, as the two decomposable $(6 \times 6)/3$ simple orthogonal multi-arrays (SOMAs) discovered by Soicher (2012a). We now define SOMA and decomposable SOMA. A SOMA(k, n), where $k \geq 0$ and $n \geq 2$ are integers, according to Soicher (2013) and Soicher (1999), is an $(n \times n)/k$ semi-Latin square which has the property that any two distinct symbols occur together in at most one entry. Let T be a SOMA. If there exist integers, $u > 0$ and $v > 0$, such that T is of type (u, v), then, T is decomposable; otherwise, T is indecomposable (Soicher, 1999).

Bailey (1992) presented the *A*, *D* and *E* efficiency values of a hypothetical Trojan square with thirty-six blocks of size three and the *MV* efficiency value was identified as the *minimum simple* efficiency measure. The difference between each of the *A*, *D*, *E* and *MV* efficiency values of the semi-Latin square in Figure 10 and each corresponding efficiency value of the hypothetical Trojan square in Table 5 of Bailey (1992) are, respectively, 0.0017, 0.0006, 0.0687 and 0.0081. One could easily see that the efficiency values of Figure 10 are almost the same as those of the hypothetical $(6 \times 6)/3$ Trojan square. Moreover, Figure 10 has exactly the same *A*, *D*, *E* and *MV* efficiency values with the $(6 \times 6)/3$ semi-Latin square of Table 5 in Bailey (1992) and which, according to Bailey (1992), compares with the hypothetical Trojan square. According to Bailey (1992), the semi-Latin square of Table 5 was constructed by Peter Wilde for use in a 1990 trial at Brooms Barn by superimposing Figure 12 on Figure 13 in Bailey (1992). Therefore, the semi-Latin square presented in Figure 10 now does not only exist but is efficient with respect to the four efficiency measures.

0 6 7	1 11 16	2 8 13	3 10 15	4 12 17	5 9 14
1 10 17	0 8 9	4 7 15	2 14 16	5 11 13	3 6 12

2	9	12	4	6	14	0	10	11	5	7	17	3	8	16	1	13	15
3	11	14	2	15	17	5	6	16	0	12	13	1	7	9	4	8	10
4	13	16	5	10	12	3	9	17	1	6	8	0	14	15	2	7	11
5	8	15	3	7	13	1	12	14	4	9	11	2	6	10	0	16	17

Figure 10: The Efficient $(6 \times 6)/3$ semi-Latin square

Furthermore, the *A*, *D*, *E* and *MV* values of the $(6 \times 6)/3$ semi-Latin squares constructed in this work were categorized and compared according to the type of $(6 \times 6)/2$ semi-Latin squares, Δ_1, Δ_2, Δ_3, Δ_4, Δ_5, Δ_6 and Δ_7, from which they are obtained. The categorization and comparisons enabled the identification of the best and worst $(6 \times 6)/3$ semi-Latin squares associated to each of the seven $(6 \times 6)/2$ semi-Latin squares, used here for construction, based on their efficiencies. The results are presented in Tables 1, 2, 3 and 4, respectively, for *A*, *D*, *E* and *MV*.

Table 1: *A*-Efficiency Values

		Number of Designs Constructed from						
S/N	***A***	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6	Δ_7
1	0.59	19	0	0	0	1	0	1
2	0.60	71	2	1	0	19	0	24
3	0.61	259	11	35	0	144	15	142
4	0.62	766	111	168	0	698	154	701
5	0.63	1622	704	675	400	1738	857	1876
6	0.64	2842	2626	2417	2140	3750	2812	3561
7	0.65	3102	4086	4243	4438	2601	3954	2593
8	0.66	723	1795	1739	2047	448	1527	500
9	0.67	4	73	130	340	9	89	10
10	0.68	0	0	0	15	0	0	0
11	0.69	0	0	0	1	0	0	0

The $(6 \times 6)/2$ semi-Latin squares, Δ_2, Δ_4, Δ_3 and Δ_6, gave the largest number of $(6 \times 6)/3$ semi-Latin squares in Table 1 with $A \geq 0.65$ where the only $(6 \times 6)/3$ semi-Latin square with $max(A)$ was constructed from Δ_4. On the other hand, the only $(6 \times 6)/3$ semi-Latin squares with $min(A)$ were constructed from Δ_1, Δ_5 and Δ_7. Note that $min(.)$ and $max(.)$ are used here to represent the *minimum* and *maximum* efficiency measures, respectively, obtained in this work. Therefore, the best $(6 \times 6)/3$ semi-Latin square in Figure 10 was constructed from Δ_4 while the worst squares were constructed from Δ_1, Δ_5 and Δ_7. From Table 2, the values of *D* of the $(6 \times 6)/3$ semi-Latin squares constructed from Δ_2, Δ_3 and Δ_6 ranged from 0.65 to 0.69. The range of efficiency values for the $(6 \times 6)/3$ semi-Latin squares constructed from Δ_4 is $D \geq 0.67$ and the only semi-Latin square with $max(D)$ (Figure 10) is constructed from Δ_4.

Table 2: D-Efficiency Values

S/N	D	Number of Designs Constructed from						
		Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6	Δ_7
1	0.65	6	0	0	0	2	0	3
2	0.66	367	43	86	0	331	45	396
3	0.67	3660	2212	2120	1730	4610	2491	4544
4	0.68	5349	6971	6996	7193	4450	6719	4441
5	0.69	26	182	206	484	15	153	24
6	0.70	0	0	0	1	0	0	0

Table 3: *E*-Efficiency Values

S/N	E	Number of Designs Constructed from						
		Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6	Δ_7
1	0.20	8	0	0	0	0	0	0
2	0.21	6	0	0	0	0	0	0
3	0.22	46	0	0	0	0	0	0
4	0.23	61	0	0	0	0	0	0
5	0.24	131	0	0	0	0	0	0
6	0.25	229	0	0	0	0	0	0
7	0.26	338	0	0	0	0	0	0
8	0.27	377	0	0	0	0	0	4
9	0.28	535	2	8	0	8	0	17
10	0.29	577	32	54	0	135	15	84
11	0.30	661	35	181	0	490	113	273
12	0.31	646	216	235	100	927	382	750
13	0.32	542	451	296	210	341	398	1278
14	0.33	689	395	398	20	701	633	674
15	0.34	774	662	612	240	1087	871	976
16	0.35	796	1024	880	970	1309	1156	1137
17	0.36	725	1371	1244	1070	1341	1301	1260
18	0.37	695	1373	1341	1391	1117	1315	1196
19	0.38	600	1217	1288	2081	887	1155	839
20	0.39	451	990	1162	1031	499	907	460
21	0.40	308	725	758	739	276	524	266
22	0.41	124	443	497	570	162	286	114
23	0.42	70	269	223	499	74	186	46
24	0.43	14	131	131	363	41	107	23
25	0.44	5	45	58	73	7	47	7
26	0.45	0	13	25	50	6	11	4
27	0.46	0	14	12	0	0	0	0
28	0.47	0	0	3	0	0	0	0
29	0.48	0	0	1	0	0	1	0
30	0.49	0	0	1	0	0	0	0
31	0.60	0	0	0	1	0	0	0

The classification based on the values of E in Table 3 indicates that only the $(6 \times 6)/3$ semi-Latin squares from Δ_1 have efficiency values ranging from $min(E)$ to 0.44. The only semi-Latin square with $max(E)$ is the design in Figure 10 which is constructed from Δ_4. The only $(6 \times 6)/3$ semi-Latin square with $max(MV)$ was constructed from Δ_4 while those of $min(MV)$ were constructed from Δ_7, followed by Δ_1. In general, the best of the $(6 \times 6)/3$ semi-Latin square, obtained in this study, was constructed from Δ_4 while the worst of the semi-Latin squares were constructed from Δ_1 and Δ_7. The poor performance of Δ_1, the Howell design, was also pointed out by Bailey and Royle (1997) which explains its absence from their *league table* of best three efficient designs with respect to each of the four statistical efficiency measure.

The efficient semi-Latin square of Figure 10 with canonical efficiency factors, 0.5980 with multiplicity three, 0.6667 with multiplicity ten, 0.8464 with multiplicity three and 1.0 with multiplicity one, is a simple orthogonal multi-array (SOMA) of order six; specifically denoted as SOMA(3, 6). Figure 10 displayed the property that no two distinct symbols appeared more than once together. As stated earlier and according to Bailey and Royle (1997), a semi-Latin square which maximizes any of the statistical efficiency measures among semi-Latin squares of that size is said to be optimal with respect to the efficiency criteria (Soicher, 2013). For instance, a semi-Latin square which maximizes A is said to be A-optimal. The $(6 \times 6)/3$ semi-Latin square presented in Figure 10 maximized *A*, *D*, *E* and *MV* among the 65856 semi-Latin squares of side six and blocks of size three and is therefore, *A*-optimal, *D*-optimal, *E*-optimal and *MV*-optimal.

Table 4: *MV*-Efficiency Values

S/N	*MV*	Number of Designs Constructed from						
		Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6	Δ_7
1	0.48	0	0	0	0	0	0	4
2	0.49	14	0	0	0	0	0	0
3	0.50	114	0	0	0	0	0	37
4	0.51	226	8	0	2	0	0	214
5	0.52	325	37	0	12	58	2	134
6	0.53	268	29	0	34	158	11	45
7	0.54	340	65	106	32	108	40	152
8	0.55	406	210	182	172	234	154	586
9	0.56	722	558	543	357	823	685	1243
10	0.57	1046	1043	888	1032	1465	1307	1934
11	0.58	1527	1595	2169	2036	1615	2499	2048
12	0.59	1589	1991	2401	1927	2172	1982	1582
13	0.60	1570	1601	1647	1721	1588	1137	779
14	0.61	929	1249	953	1091	651	818	455
15	0.62	295	772	341	700	432	540	156
16	0.63	37	195	116	201	88	185	37
17	0.64	0	50	46	85	15	48	2
18	0.65	0	5	16	5	1	0	0
19	0.66	0	0	0	1	0	0	0

4. Conclusion

The method of superposition was used to construct efficient $(6 \times 6)/3$ semi-Latin square by superimposing each reduced Latin square of degree six on each of the $(6 \times 6)/2$ semi-Latin squares in context. The most efficient $(6 \times 6)/3$ semi-Latin square identified in this study and given in Figure 10 has the same *A*, *D*, *E* and *MV* statistical efficiency values with the two efficient $(6 \times 6)/3$ semi-Latin squares developed by Soicher (2012b). The $(6 \times 6)/3$ semi-Latin square in Figure 10 is also *A*-, *D*-, *E*- and *MV*-optimal and is a SOMA(3,6), a special class of semi-Latin squares used in the design of experiments (Soicher, 1999).

With minimum and maximum efficiency values of (0.59, 0.69) and (0.65, 0.70), respectively, for *A* and *D*, the *A* and *D* efficiency measures, the $(6 \times 6)/3$ semi-Latin squares have very low variability while the *E* efficiency measure varies greatly with respect to the minimum and maximum efficiency values, (0.20, 0.60). Bailey and Royle (1997) pointed out that *A* and *D* of the $(6 \times 6)/2$ semi-Latin squares have very low variability because they are the means of the canonical efficiency factors whose arithmetic mean is fixed while *E* is an extreme value. The low variability of *A* is important since *A* is paramount in the estimate of treatment effects and since *A* measures the inverse of the average variance of all normalized contrasts where minimizing the variance is required. The *D* efficiency, which measures the inverse of the volume of the confidence ellipsoid, is relevant in experiments involving tests of models and sub-models. Since decisions are made based on the results of such tests, the low variability of *D* is important. The high variability of *E* is useful only if decision needs to be made about a contrast which is not known in advance and if the contrast turns out to be the one with the highest variance.

References

Bailey, R.A. (1992). Efficient semi-Latin squares, *Statistica Sinica*, 2, 413-437.

Bailey, R.A. (1997). A Howell design admitting A_5^1, *Discrete Mathematics*, 167/168, 65 – 71.

Bailey, R.A. (2011). Symmetric factorial designs in blocks, *Journal of Statistical Theory and Practice*, 5(1), 13 – 24.

Bailey, R.A. and Chigbu, P.E. (1997). Enumeration of semi-Latin squares, *Discrete Mathematics*, 167/168, 73 – 84.

Bailey, R.A. and Royle, G. (1997). Optimal semi-Latin squares with side six and block size two, *Proceedings of the Royal Society London A*, 453 (1964), 1903 – 1914.

Bailey, R.A. (1990). An Efficient semi-Latin square for twelve treatments in blocks of size two, *Journal of Statistical Planning and Inference*, 26, 263-266.

Chigbu, P.E., Mba, E.I. and Ukaegbu, E.C. (2021). Automating the construction and evaluation of eighteen-treatment semi-Latin squares of order six in blocks of size three. *Journal of the Nigerian Statistical Association*, 33, 80 – 99.

Denes, J. and Keedwell, A.D. (1991). Latin squares - new developments in the theory and applications, *Annals of Discrete Mathematics*, 46, 1 – 453.

Hung, S.H.Y. and Mendelsohn, N.S. (1974). On Howell designs, *Journal of Combinatorial Theory*, A 16, 174 – 198.

Kreher, D.L., Royle, G.F. and Wallis, W.D. (1996). A family of resolvable regular graph designs, *Discrete Mathematics*, 156, 269 – 275.

McKay, B.D. (2019). Reduced Latin squares, McKay's Personal webpage. Available at www.users.cecs.anu.au/~bdm/.

McKay, B.D. and Wanless, I.M. (2005). On the number of Latin squares. *Annals of Combinatorics*, 9, 335 – 344.

Meszka, M. & Tyniec, M. (2019). Orthogonal one-factorizations of complete multipartite graphs, *Designs, Codes and Cryptography*, 87, 987 – 993.

Phillips, N.C.K. and Wallis, W.D. (1996). All solutions to a tournament problem, *Congressum Numerantium*, 114, 193 – 196.

Preece, D.A. and Philips, N.C.K. (2002). Euler at the bowling green, *Utilitas Mathematica*, 61, 129 – 165.

Seah, E. and Stinson, D.R. (1987). An assortment of new Howell designs, *Utilitas Mathematica*, 31, 175 – 188.

Soicher, L.H. (2013). Optimal and efficient semi-Latin squares, *Journal of Statistical Planning and Inference*, 143, 573 – 582.

Soicher, L.H. (1999). On the structure and classification of SOMAs: Generalizations of mutually orthogonal Latin squares. *Electronic Journal of Combinatorics*, 6 (1), R32. Available at www.elibm.org/article/10002268.

Soicher, L.H. (2012a). SOMA Update, http://www.maths.qmul.ac.uk/~leonard/soma.

Soicher, L.H. (2012b). Uniform semi-Latin squares and their Schur-optimality, *Journal of Combinatorial Design*, 20, 265 – 277.